SHARPNESS OF RICKMAN'S PICARD THEOREM IN ALL DIMENSIONS

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Abstract. We show that given $n \geq 3$, $q \geq 1$, and a finite set $\{y_1, \ldots, y_q\}$ in $\mathbb{R}^n$ there exists a quasiregular mapping $\mathbb{R}^n \to \mathbb{R}^n$ omitting exactly points $y_1, \ldots, y_q$.

1. Introduction

By the classical Picard theorem an entire holomorphic map $\mathbb{C} \to \mathbb{C}$ omits at most one point if non-constant. The characteristic example of an entire holomorphic map omitting a point is, of course, the exponential function $z \mapsto e^z$, since every entire holomorphic map $\mathbb{C} \to \mathbb{C}$ omitting a point factors through the exponential map.

Liouville's theorem asserts that all entire conformal maps $\mathbb{R}^n \to \mathbb{R}^n$ are Möbius transformations and, in particular, homeomorphisms for $n \geq 3$. This rigidity of spatial conformal geometry no longer persists in quasiconformal geometry. Reshetnyak in the late 1960's and Martio–Rickman–Väisälä in the early 1970's showed that the rich theory of *mappings of bounded distortion*, or so-called *quasiregular mappings*, is a natural replacement for holomorphic functions in higher dimensions. This advancement raised the question of the existence of Picard type theorems for quasiregular mappings; see e.g Zorich [20] or Väisälä's survey [18].

Already in his 1967 paper [20] Zorich gave an example of a quasiregular mapping $\mathbb{R}^n \to \mathbb{R}^n$ omitting the origin. This so-called *Zorich map* is the natural higher-dimensional analog of the exponential function although the mapping is not a local homeomorphism. The branching of the map cannot be avoided by Zorich's *Global Homeomorphism Theorem* from the same article: *For $n \geq 3$, quasiregular local homeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ are homeomorphisms.* Recall that by Reshetnyak's theorem quasiregular mappings are (generalized) branched covers, that is, discrete and open mappings and hence local homeomorphisms modulo an exceptional set of (topological) codimension 2; we refer to Rickman's monograph [14] for the general theory of quasiregular mappings.

A counterpart of Picard's theorem for quasiregular mappings is due to Rickman [12]: *Given $K > 1$ and $n \geq 2$ there exists q depending only on K and n so that a non-constant K-quasiregular mapping $\mathbb{R}^n \to \mathbb{R}^n$ omits at most q points.* The sharpness of Rickman's Picard theorem is known in dimension $n = 3$ and is also due to Rickman. In [13] he shows the following existence result: *Given any finite set P in $\mathbb{R}^3$ there exists a quasiregular mapping $\mathbb{R}^3 \to \mathbb{R}^3$ omitting exactly P.*

Holopainen and Rickman generalized the Picard theorem to quasiregular mappings into manifolds with many ends in [4] and *a fortiori* to quasiregular mappings between manifolds in [6]; note also similar results in the sub-Riemannian geometry [5]. These result stem from potential theoretic proofs of Rickman's Picard theorem due to Lewis [7] and Eremenko–Lewis [2]. It can be said that the ramifications of these methods are now well-understood. Recently, Rajala generalized Rickman's

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Picard theorem to mappings of finite distortion [11]. Whereas the aforementioned potential-theoretic methods are difficult to adapt to this more general class of mappings, Rajala shows that value distribution theory based on modulus methods is still at our disposal.

The sharpness of these theorems, however, is still mostly unknown and Rickman's three-dimensional construction in [13] provides essentially the only method to produce examples.

In this article we show the precision of Rickman's Picard theorem in all dimensions.

Theorem 1.1. *Given $n \geq 3$, $q \geq 2$, and points $y_1, \ldots, y_q$ in $\mathbb{R}^n$ there exists a quasiregular mapping $\mathbb{R}^n \to \mathbb{R}^n$ omitting exactly points $y_1, \ldots, y_q$.*

It has already been mentioned that the case of dimension $n = 3$ was settled by Rickman. For $n = 2$ the number of omitted points is at most 1 by Picard's theorem and the Stoïlow factorization; see e.g. book of Astala, Iwaniec, and Martin [1, Section 5.5]. As discussed above, the case $q = 1$ is given by the Zorich map for all $n \geq 3$. Therefore we may restrict to cases $n \geq 4$ and $q \geq 2$. However, it is natural to include $n = 3$.

As will become apparent in the following outline of the proof, the proof of Theorem 1.1 is independent of the analytic theory of quasiregular mappings.

The general outline follows the idea of Rickman's construction in [13] and both proofs stem from PL-topology. Rickman's original method, however, relies on a very delicate deformation theory of 2-dimensional branched covers ([13, Section 5]) which leads to an extension theory of 2-dimensional branched covers. These arguments rely essentially on the discreteness nature of the branch set in dimension 2. Already when $n = 3$, the corresponding deformation theory is much more complicated due to the non-trivial topology of the branch set; see however application of Piergallini's result in [10] to construct a quasiregular map $\mathbb{R}^4 \to \mathbb{S}^2 \times \mathbb{S}^2 \# \mathbb{S}^2 \times \mathbb{S}^2$ in [15]. We are not aware of similar deformation theory, based on a detailed analysis of the branch set, in higher dimensions.

It turns out, however, that the required extension theory is essentially trivial in all dimensions for BLD-mappings. Recall that a mapping $f\colon X \to Y$ between metric spaces X and Y is a *mapping of bounded length distortion* (or a *BLD-map*, for short) if f is open and discrete, and there exists a constant $L \geq 1$ satisfying

$$\frac{1}{L}\ell(\gamma) \leq \ell(f \circ \gamma) \leq L\ell(\gamma) \tag{1.1}$$

for all paths γ in X, where $\ell(\gamma)$ is the length of γ. We refer to the seminal paper of Martio and Väisälä [9] for the discussion of the special rôle of BLD-mappings among quasiregular mappings; see also Heinonen–Rickman [3] for the metric theory.

The BLD-theory in the proof of Theorem 1.1 brings forth an alternative, and slightly stronger, formulation. We denote by $\mathbb{S}^n$ and $\mathbb{S}^{n-1}$ the Euclidean unit spheres in $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, respectively, and by $B^n(y, \delta)$ the metric ball in $\mathbb{S}^n$ in the inherited metric.

Theorem 1.2. *Let $n \geq 3$, $p \geq 2$, and $y_0, \ldots, y_p$ be points in $\mathbb{S}^n$. Let also g be a Riemannian metric on $M := \mathbb{S}^n \setminus \{y_0, \ldots, y_p\}$ for which $B^n(y_i, \delta) \setminus \{y_i\}$ is isometric, in metric g, to $\mathbb{S}^{n-1}(\delta) \times (0, \infty)$ for some $\delta > 0$ and all $0 \leq i \leq p$. Then there exists a surjective BLD-mapping $\mathbb{R}^n \to (M, g)$.*

Theorem 1.2 clearly yields Theorem 1.1 as a corollary. Indeed, let $y_1, \ldots, y_q$ be points in $\mathbb{R}^n$. After identifying $\mathbb{R}^n$ with $\mathbb{S}^n \setminus \{e_{n+1}\}$ by stereographic projection, we may fix a Riemannian metric g on $M := \mathbb{S}^n \setminus \{e_{n+1}, y_1, \ldots, y_q\}$ and a BLD-mapping $f\colon \mathbb{R}^n \to (M, g)$ as in Theorem 1.2. It is now easy to verify that the identity map

$(M,g) \to \mathbb{S}^n \setminus \{e_{n+1}, y_1, \ldots, y_q\}$ is quasiconformal. Thus $f\colon \mathbb{R}^n \to \mathbb{R}^n \setminus \{y_1, \ldots, y_q\}$ is quasiregular.

We are not aware of other methods of producing examples of BLD-mappings from $\mathbb{R}^n$ into Riemannian manifolds with many ends.

1.1. **Outline of the proof.** Using the framework of Theorem 1.2, we outline the construction of a BLD-map $F\colon \mathbb{R}^n \to \mathbb{S}^n \setminus \{y_0, \ldots, y_p\}$ for $p > 2$, and again identify $\mathbb{R}^n$ with $\mathbb{S}^n \setminus \{e_{n+1}\}$ by stereographic projection. It is no restriction to assume that $y_0 = e_{n+1}$ and $y_i = (0, t_i) \in \mathbb{R}^{n-1} \times \mathbb{R} \subset \mathbb{S}^n$ for $-1 < t_1 < t_2 < \ldots < t_p < 1$ and we will assume so from now on.

Setting aside geometric aspects of the construction, we give first the topological description of $F\colon \mathbb{R}^n \to \mathbb{S}^n \setminus \{y_0, \ldots, y_p\}$. This description is based on certain essential partitions of $\mathbb{R}^n$ and $\mathbb{S}^n$. Given a closed set X in $\mathbb{R}^n$ (or in $\mathbb{S}^n$), we say that a finite collection of closed sets $X_1, \ldots, X_m$ form an *essential partition of X* if $X_1 \cup \cdots \cup X_m = X$ and sets X_i have pair-wise disjoint interiors.

In the target $\mathbb{S}^n \setminus \{y_0, \ldots, y_p\}$, we fix an essential partition $E_0, \ldots, E_p$ of $\mathbb{S}^n$ into n-cells, with $y_i \in E_i$ for $0 \le i \le p$, so that $E_0 = \mathbb{S}^n \setminus B^n$ and $E_1 \cup \cdots \cup E_p = \bar{B}^n$. We also assume that $E_{i-1} \cap E_i \cap E_{i+1} = \mathbb{S}^{n-2}$ and $E_i \cap E_{i+1}$ is an $(n-1)$-cell for all $i \pmod{p+1}$; see Figure 1, and denote $\mathbf{E} = (E_0, \ldots, E_p)$.

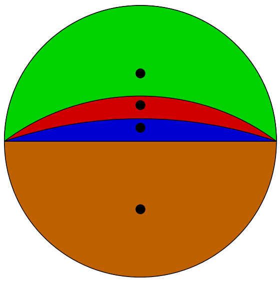

FIGURE 1. Cells $E_1, \ldots, E_4$ with (marked) points $y_1, \ldots, y_4$ for $p = 4$ (and $n = 2$).

The F-induced essential partition of $\mathbb{R}^n$ is more complicated. Let $E' \subset \mathbb{S}^n$ be a closed set satisfying $E' = \mathrm{cl}(\mathrm{int}E')$. We say that a mapping $\varphi\colon \bar{B}^n \to E'$ is a *branched cover modulo boundary* if $\varphi|B^n\colon B^n \to \mathrm{int}E'$ is a branched cover and, for every branched cover $\psi\colon \partial E' \to \mathbb{S}^{n-1}$, the mapping $\psi \circ \varphi|\partial B^n\colon \partial B^n \to \mathbb{S}^{n-1}$ is a branched cover. Furthermore, we say that E' is an *n-cell modulo boundary* if there exists a branched cover modulo boundary $\varphi\colon \bar{B}^n \to E'$ which is a homeomorphism in the interior, that is, $\varphi|B^n\colon B^n \to \mathrm{int}E'$ is a homeomorphism. Note that $\partial E'$ need not be homeomorphic to $\mathbb{S}^{n-1}$; see Figure 2.

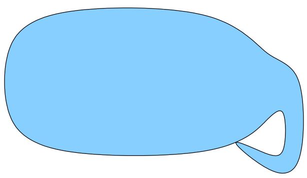

FIGURE 2. A 2-cell modulo boundary.

Suppose, for the sake of argument, there is an essential partition $E'_0, \ldots, E'_p$ of $\mathbb{S}^n$ into compact sets so that $E'_0 \cap \cdots \cap E'_p = \{e_{n+1}\}$ and each E'_i has an essential partition $E'_{i,1}, \ldots, E'_{i,j_i}$ into n-cells modulo boundary.

Having these essential partitions at our disposal, we reduce first the existence of a branched cover $F\colon \mathbb{R}^n \to \mathbb{S}^n \setminus \{y_0, \ldots, y_p\}$ to that of the existence of a branched cover $f\colon \partial_\cup \mathbf{E}' \to \partial_\cup \mathbf{E}$, where $\mathbf{E}' = (E_0', \ldots, E_p')$. Here, and in what follows, the notation

$$\partial_\cup \mathbf{X} = \partial X_0 \cup \cdots \cup \partial X_p$$

is used whenever $\mathbf{X} = (X_0, \ldots, X_p)$ is an essential partition.

Suppose $f\colon \partial_\cup \mathbf{E}' \to \partial_\cup \mathbf{E}$ is a branched cover satisfying the additional condition $f(\partial E_{i,j}) = \partial E_i$ for every $i = 0, \ldots, p$ and $1 \le j \le j_i$. Since $E_{i,j}'$ is an n-cell modulo boundary and E_i is an n-cell, we observe that each branched cover $f_{i,j} = f|\partial E_{i,j}$ extends to a branched cover $F_{i,j}\colon E_{i,j}' \setminus \{e_{n+1}\} \to E_i \setminus \{y_i\}$. Indeed, we may fix, for every i and j, a branched cover modulo boundary $\varphi_{i,j}\colon \mathbb{R}^{n-1} \times [0,\infty) \to E_{i,j}' \setminus \{e_{n+1}\}$, which is a homeomorphism in the interior, as well as a homeomorphism $\psi_i\colon \mathbb{S}^{n-1} \times [0,\infty) \to E_i \setminus \{y_i\}$. This means that $h_{i,j} = \psi_i^{-1} \circ f_{i,j} \circ \varphi_{i,j}|\mathbb{R}^{n-1}\colon \mathbb{R}^{n-1} \to \mathbb{S}^{n-1}$ is a branched cover. The (trivial) extension $h_{i,j} \times \mathrm{id}\colon \mathbb{R}^{n-1} \times [0,\infty) \to \mathbb{S}^{n-1} \times [0,\infty)$ of $h_{i,j}$ now yields the required extension of $f_{i,j}$ after pre- and post-composition with ψ_i and $\varphi_{i,j}^{-1}$, respectively. Thus f *extends to a branched cover* $F\colon \mathbb{S}^n \setminus \{e_{n+1}\} \to \mathbb{S}^n \setminus \{y_1, \ldots, y_p\}$.

Observe also that in forthcoming constructions we may view $\partial_\cup \mathbf{E}'$ and $\partial_\cup \mathbf{E}$ as branched codimension-1 hypersurfaces in $\mathbb{R}^n$ and the map f as a (generalized) Alexander map. In particular, the Zorich map is of this character when $p = 2$.

It is crucial that this simple extension is also available for BLD-mappings. Let $\Omega_i = E_i' \setminus \{e_{n+1}\}$ and $\Omega_{i,j} = E_{i,j}' \setminus \{e_{n+1}\}$ for all i and j, so that $\mathbf{\Omega} = (\Omega_1, \ldots, \Omega_p)$ is an essential partition of $\mathbb{R}^n$. It is now a simple exercise to observe that the extension $F\colon \mathbb{R}^n \to \mathbb{S}^n \setminus \{y_0, \ldots, y_p\}$ constructed above will be a BLD-mapping with respect to Riemannian metric g in $\mathbb{S}^n \setminus \{y_0, \ldots, y_p\}$ if

(i) $f\colon \partial_\cup \mathbf{\Omega} \to \partial_\cup \mathbf{E}$ is a BLD-map,
(ii) $\varphi_i\colon \mathbb{R}^{n-1} \times [0,\infty) \to \Omega_i$ is BLD modulo boundary and $\varphi_i|\mathbb{R}^{n-1} \times (0,\infty)$ is an embedding,
(iii) $\psi_i\colon \mathbb{S}^{n-1} \times [0,\infty) \to (E_i \setminus \{y_i\}, g)$ is bilipschitz.

Here and in what follows, we say that a mapping $\varphi\colon \mathbb{R}^{n-1} \times [0,\infty) \to \Omega$, where Ω is a closed set in $\mathbb{R}^n$ with $\Omega = \mathrm{cl}(\mathrm{int}\Omega)$, is *BLD modulo boundary* if the restriction $f|\mathbb{R}^{n-1} \times (0,\infty)\colon \mathbb{R}^{n-1} \times (0,\infty) \to \mathrm{int}\Omega$ is BLD, and for every BLD-map $\psi\colon \partial\Omega \to \mathbb{S}^{n-1}$, the map $\psi \circ \varphi|\mathbb{R}^{n-1} \times \{0\}\colon \mathbb{R}^{n-1} \times \{0\} \to \mathbb{S}^{n-1}$ is BLD.

For Riemannian metrics g with cylindrical ends as in Theorem 1.2, it is easy to construct homeomorphisms ψ_i satisfying condition (iii), and so this extension argument reduces the proof of Theorem 1.2 to Theorem 1.3.

A closed set Ω in $\mathbb{R}^n$ is a *Zorich extension domain* if there exists a map $\varphi\colon \mathbb{R}^{n-1} \times [0,\infty) \to \Omega$ which is BLD modulo boundary and a homeomorphism in the interior.

Theorem 1.3. *Given $n \ge 3$ and $p \ge 2$ there is an essential partition $\mathbf{\Omega} = (\Omega_0, \ldots, \Omega_p)$ of $\mathbb{R}^n$ for which*

(a) *the sets Ω_i have essential partitions into Zorich extension domains, and*
(b) *there exists a BLD-map $f\colon \partial_\cup \mathbf{\Omega} \to \partial_\cup \mathbf{E}$ satisfying $f(\partial \Omega_i) = \partial E_i$ for all $i = 0, \ldots, p$.*

Essential partitions satisfying both conditions (a) and (b) in Theorem 1.3 are called *Rickman partitions*.

This partition is achieved in two stages, using rough Rickman partitions; an essential partition $\widetilde{\mathbf{\Omega}} = (\widetilde{\Omega}_0, \ldots, \widetilde{\Omega}_p)$ of $\mathbb{R}^n$ is a *rough Rickman partition* if

(a′) each $\widetilde{\Omega}_i$ has an essential partition $(\widetilde{\Omega}_{i,1}, \ldots, \widetilde{\Omega}_{i,j_i})$ with each $\widetilde{\Omega}_{i,j}$ BLD-homeomorphic to $\mathbb{R}^{n-1} \times [0,\infty)$, and

(b′) the sets $\partial_\cup \widetilde{\mathbf{\Omega}}$ and $\partial_\cap \widetilde{\mathbf{\Omega}}$ have finite Hausdorff distance, where

$$\partial_\cap \widetilde{\mathbf{\Omega}} = \bigcap_i \widetilde{\Omega}_i$$

is the *common boundary of the partition* $\widetilde{\mathbf{\Omega}}$; $\partial_\cup \widetilde{\mathbf{\Omega}}$ is called the *pair-wise common boundary of* $\widetilde{\mathbf{\Omega}}$.

Rough Rickman partitions $\widetilde{\mathbf{\Omega}}$ do not admit branched covers $\partial_\cup \widetilde{\mathbf{\Omega}} \to \partial_\cup \mathbf{E}$ in general. To refine our rough Rickman partition $\widetilde{\mathbf{\Omega}}$ to a Rickman partition $\mathbf{\Omega}$, we show that there exist rough Rickman partitions fulfilling an additional compatibility condition, the so-called tripod property; see Definition 4.4 for its precise formulation. These particular rough Rickman partitions, together with a modification of Rickman's *sheet construction* in [13, Section 7], yield the required global partition $\mathbf{\Omega}$.

In Rickman's original terminology, the construction of rough Rickman partitions is called the *cave construction* and the notion of *cave bases* corresponds to the subdivisions provided by the tripod property. The reader may find it interesting to compare the procedure here with [13, Sections 2 and 3].

We summarize the two parts of the proof of Theorem 1.2 as follows. First, we prove the existence of suitable rough Rickman partitions by direct construction.

Proposition 1.4. *Given $n \geq 3$ and $p \geq 2$ there exists a rough Rickman partition $\widetilde{\mathbf{\Omega}} = (\widetilde{\Omega}_0, \ldots, \widetilde{\Omega}_p)$ supporting the tripod property.*

As in [13] we consider first the case $p = 2$ and obtain a rough Rickman partition $\widetilde{\mathbf{\Omega}}' = (\Omega_0', \Omega_1', \Omega_2')$ with Ω_0' and Ω_1' BLD-homeomorphic to $\mathbb{R}^{n-1} \times [0, \infty)$ and Ω_2' having an essential partition $(\Omega_{2,1}', \ldots, \Omega_{2,2^n}')$, where each $\Omega_{2,j}'$ is BLD-homeomorphic to $\mathbb{R}^{n-1} \times [0, \infty)$. All sets Ω_i' are unions of unit n-cubes $[0, 1]^n + v$ where $v \in \mathbb{Z}^n$, and $\widetilde{\mathbf{\Omega}}'$ satisfies the tripod property. This occupies Section 5. The final step of the general proof of Proposition 1.4 is a partition of Ω_2' by an essential partition $(\Omega_2, \ldots, \Omega_p)$ so that $(\Omega_0', \Omega_1', \Omega_2, \ldots, \Omega_p)$ produes a rough Rickman partition. This step is discussed in Section 8.

The set $\widetilde{\mathbf{\Omega}}'$ (and similarly $\widetilde{\mathbf{\Omega}}$) has the following geometric properties. Let X be any of the sets Ω_0', Ω_1' or $\Omega_{2,j}'$ for some $1 \leq j \leq 2^n$, and for each $k \geq 0$ write

$$X_k = 3^{-k} X.$$

Thus upon passing to a subsequence if necessary, the sets X_k and their boundaries $\partial X_k \subset \mathbb{S}^n$ converge in the Hausdorff sense respectively to X_∞ and ∂X_∞, where ∂X_∞ is a "generalized Alexander horned sphere in $\mathbb{S}^n$ with infinitely many horns." Under the normalization $\widetilde{\mathbf{\Omega}}_k = 3^{-k} \widetilde{\mathbf{\Omega}}$ for $k \geq 0$, in fact $\partial_\cup \mathbf{\Omega}_\infty = \partial_\cap \mathbf{\Omega}_\infty$ for any sublimit $\mathbf{\Omega}_\infty$ of the partitions $\widetilde{\mathbf{\Omega}}_k$, in the Hausdorff sense. This may be interpreted as a *coarse Lakes of Wada* property for the pair-wise common boundary of $\widetilde{\mathbf{\Omega}}$. Of course, both observations apply to Rickman's original cave construction. We do not discuss these geometric properties of the partition $\widetilde{\mathbf{\Omega}}$ in more detail, and leave these details to the interested reader.

The second part of the proof of Theorem 1.3 is the refinement of rough Rickman partitions to Rickman partitions. This formalizes the effect of the sheet construction (called *pillows* in Section 7) as follows.

Proposition 1.5. *Given a rough Rickman partition $\widetilde{\mathbf{\Omega}} = (\widetilde{\Omega}_0, \ldots, \widetilde{\Omega}_p)$ supporting the tripod property there exists a Rickman partition $\mathbf{\Omega} = (\Omega_0, \ldots, \Omega_p)$ for which the Hausdorff distance of $\partial_\cup \mathbf{\Omega}$ and $\partial_\cup \widetilde{\mathbf{\Omega}}$ is at most* 1.

As discussed in this introduction, Propositions 1.4 and 1.5 together prove Theorem 1.3, and we obtain Theorem 1.2 from Theorem 1.3 and the observation on the existence of BLD-extensions.

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Contents

2. Preliminaries

In this section we discuss general metric and combinatorial notions in the construction. Most of the discussion is in the ambient space $\mathbb{R}^n$ for some fixed $n \geq 3$.

2.1. **Metric notions.** In $\mathbb{R}^n$, let d_∞ be the sup-metric

$$d_\infty(x,y) = \|x-y\|_\infty$$

given by the *supremum norm*

$$\|(x_1,\ldots,x_n)\|_\infty = \max_i |x_i|.$$

The metric ball $B_\infty(p,r) = \{x \in \mathbb{R}^n \colon \|p-x\|_\infty < r\}$ of radius $r>0$ about $p \in \mathbb{R}^n$ in this metric is the open cube

$$B_\infty(p,r) = p + (-r,r)^n.$$

Similarly, $\bar{B}_\infty(p,r) = p + [-r,r]^n$.

Diverting from standard terminology, we apply the term 'cube' exclusively to closed n-balls $\bar{B}_\infty(p,r)$. We say that p is the center of the cube $\bar{B}_\infty(p,r)$; the side length of $\bar{B}_\infty(p,r)$ is of course $2r$.

The set E in $\mathbb{R}^n$ is called *rectifiably connected* if for all $x,y \in E$ there exists a path $\gamma\colon [0,1] \to E$ of finite length so that $x,y \in \gamma[0,1]$. In this situation, γ *connects x and y in E.*

When E is a rectifiably connected in $\mathbb{R}^n$, d_E is its inner metric in E; that is, for all $x,y \in E$,

$$d_E(x,y) = \inf_\gamma \ell(\gamma),$$

over all paths γ connecting x and y in E, with $\ell(\gamma)$ the length of γ. Note that the length of γ is in terms of Euclidean distance. The notion of inner metric gives the following characterization of BLD-homeomorphisms: *A homeomorphisms $f\colon E \to E'$ between rectifiable sets E and E' in $\mathbb{R}^n$ is BLD if and only if $f\colon E \to E'$ is bilipschitz in the inner metric.*

2.2. **Complexes.** For a detailed discussion on simplicial complexes we refer to [16] and merely recall some notation and terminology. Given a simplicial complex P in $\mathbb{R}^n$, $P^{(k)}$ is its *k-skeleton*; that is, the collection of all k-simplices in P. If m is the largest dimension of simplices in P, then P has *dimension m*, $m = \dim P$. We consider only *homogeneous* simplicial complexes, that is, every simplex in P is contained in a simplex of dimension $\dim P$. We denote by $|P^{(k)}|$ the subset in $\mathbb{R}^n$ which is the union of all simplices in P; thus, $|P| = |P^{(m)}|$.

Recall that every k-simplex σ has a standard structure as a simplicial complex having σ as its only k-simplex and the vertices of σ as the 0-skeleton. The i-simplices of this complex form the *i-faces* of σ.

We mainly consider cubical complexes. Much as simplices have a natural structure as a complex, the k-dimensional faces of a cube $Q = \bar{B}_\infty(x,r)$ determine a natural CW complex structure for Q. The k-dimensional faces of Q are called the *k-cubes*, and a CW complex P is a *cubical complex* if its cells are cubes. Note in particular, that given i-cube Q and j-cube Q' the intersection $Q \cap Q'$ is a k-dimensional, $k \le \min\{i,j\}$, face of both cubes. The k-skeleton and its realization are defined for cubical complexes in a manner analogous to simplicial complexes.

A homogeneous cubical complex of dimension k is usually referred to as a *cubical k-complex.* A set $E \subset \mathbb{R}^n$ is a *cubical k-set* if there is a cubical k-complex P with $|P| = E$.

Two cubical k-sets E and E' are *essentially disjoint* if $E \cap E'$ is a cubical set of lower dimension. Cubical k-sets $E_1 \ldots, E_r$ induce the *essential partition $\{E_1,\ldots,E_r\}$ of a cubical set E* if $E = E_1 \cup \cdots \cup E_r$ and the sets E_i are pairwise essentially disjoint. If the sets $E_1,\ldots,E_r$, and E are n-cells, we usually consider the essential partition ordered and denote it $(E_1,\ldots,E_r)$ as in the introduction.

Finally, given two cubical sets E and E', write

$$E - E' = \mathrm{cl}(E \setminus E'),$$

where $\mathrm{cl}(E \setminus E')$ is the closure of $E \setminus E'$. Clearly, $E - E' = E$ if E' has lower dimension than E.

A cubical k-complex P is *r-fine* if all k-cubes in P have side length r, i.e. are congruent to $[0,r]^k \subset \mathbb{R}^k \subset \mathbb{R}^n$. Similarly, a set E in $\mathbb{R}^n$ is *r-fine* if $r > 0$ is the largest integer for which there exists an r-fine cubical complex P with $E = |P|$, and we call r the *side length $\rho(E)$ of E*. In what follows, we assume that all cubical complexes are r-fine for some integer $r > 0$. Given an r-fine set $E = |P|$, we tacitly assume that its underlying complex P is also r-fine.

Let P be a $3k$-regular cubical n-complex for $k \geq 1$, and $\Omega = |P|$. We denote by Ω^* the subdivision of Ω into cubes of side length 3. More formally, there exists a unique 3-fine cubical n-complex $\widetilde{P}$ satisfying $\Omega = |\widetilde{P}|$, and we denote $\Omega^* = \widetilde{P}^{(n)}$. We call Ω^* the *3-fine subdivision of* Ω. We will also need $\Omega^\#$, the *1-fine subdivision of* Ω, i.e. subdivision of Ω into unit cubes, and also refer to $\Omega^\#$ as the *unit subdivision of* Ω.

In what follows we use the notation rA for $r > 0$ and $A \subset \mathbb{R}^n$:

$$rA = \{rx \in \mathbb{R}^n : x \in A\}.$$

Given an essential partition $\mathbf{U} = (U_1, U_2, U_3)$, we denote $r\mathbf{U} = (rU_1, rU_2, rU_3)$.

2.3. **Graphs, forests, and adjacency.** We analyze the geometric and topological structure of cubical complexes and their realizations in $\mathbb{R}^n$ in terms of graphs. The pair $G = (V, E)$ is *a graph* if V is a countable set and E is a collection of unoriented pairs of points in V; V is the set of *vertices* and E the *edges* of G. Note we only allow one edge between two distinct vertices and, in particular, our graphs do not have *loops*, i.e. edges from a vertex to itself.

We use repeatedly the standard fact that a graph contains a maximal tree, that is, given a graph $G = (V, E)$ there is a subtree $T = (V, E')$ containing all vertices of G. The length $\ell(G)$ of G is the number of vertices of G, the *valence of G at v* is $\nu(G, v)$ and $\nu(G) = \max_{v \in G} \nu(G, v)$ is the *(maximal) valence of G*. We denote by $d_G(v, v')$ the graph distance of v and v' in G, that is, the length of the shortest edge path between v and v' in G.

Given a distinguished vertex $v \in G$, the pair (G, v) is called a *rooted graph* and v the *root* of this graph. The *radius $r(G, v)$ of G at v* is the largest graph distance between v and a leaf of G. A vertex $w \in G$ is a *leaf* if it belongs to exactly one edge, or equivalently, has valence 1. A vertex which is neither a leaf nor the root is an *inner vertex*. A subtree $\Gamma \subset G$ connecting the root v to a leaf w of G is a *branch* when all vertices in Γ other than v and w have valence 2.

Let (G, v) be a finite rooted tree and $v' \neq v$ a vertex in G. We define the *subtree behind v' in (G, v)* as follows. Since G is a tree, there exists unique $v'' \in G$ for which $e = \{v'', v'\}$ is the last edge in the shortest path from v to v'. The graph $(V, E \setminus \{e\})$ has two connected components Γ_v and $\Gamma_{v'}$ containing v and v', respectively. Both component are trees; $\Gamma_{v'}$ is the *subtree behind v' in (G, v)*.

A graph G is a *forest* if all of its components are trees. A forest $F \subset G$ is *maximal* if components of F are maximal trees in components of G and F contains all vertices of G.

A function $u \colon G \to \mathbb{R}$ on a tree G has the *John property in G* if given v and v' in G there exists $0 \leq j \leq d = d_G(v, v')$ so that u is (strictly) increasing on $v_0, \ldots, v_j$ and decreasing on $v_{j+1}, \ldots, v_d$, where $v = v_0, v_1, \ldots, v_d = v'$ is the unique shortest edge path from v to v' in G.

Most graphs we consider are adjacency graphs of collections of k-cells in $\mathbb{R}^n$. A set $E \subset \mathbb{R}^n$ is a *k-cell* if E is homeomorphic to the closed cube $[0,1]^k$ in $\mathbb{R}^k$; E is a *cubical k-cell* if $E = |P|$ is a k-cell, where P is an r-fine homogeneous cubical complex for $r \geq 1$.

Two k-cells E and E' are *adjacent* if $E \cap E'$ is an $(k-1)$-cell. We recall from PL theory that given two adjacent PL k-cells E and E' there exists a PL homeomorphism $E \cup E' \to E$ which is identity on $\partial(E \cup E') \cap E$, and refer to [16, Chapter 3] for this and similar results in PL theory.

A collection $\mathcal{P}$ of k-cells in $\mathbb{R}^n$ has the adjacency graph

$$\Gamma(\mathcal{P}) = (\mathcal{P}, \{\{E, E'\} \colon E \in \mathcal{P} \text{ and } E' \in \mathcal{P} \text{ are adjacent}\}).$$

When P is a cubical k-complex, we write $\Gamma(P) = \Gamma(P^{(k)})$ for short. Given a subgraph $\Gamma \subset \Gamma(P)$, we denote $|\Gamma| = \bigcup_{Q \in \Gamma} Q$; in particular, $|\Gamma(P)| = |P|$.

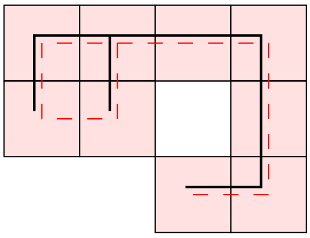

FIGURE 3. A cubical 2-complex with its adjacency graph and a choice of a maximal tree.

2.4. **Remarks on figures.** Although we consider n-cells for $n \geq 3$, we use two-dimensional illustrations related to three-dimensional example configurations.

In particular, 'fold-out' diagrams illustrate particular cubical $(n-1)$-complexes. To formalize this, suppose E is a cubical $(n-1)$-cell in $\mathbb{R}^n$ with an essential partition $\{E_1, \ldots, E_s\}$ into unit $(n-1)$-cubes and let Γ be a maximal tree in $\Gamma(\{E_1, \ldots, E_s\})$. We say that an $(n-1)$-cell E' in $\mathbb{R}^{n-1}$ is a *fold-out of E (along Γ)* if E' has a partition $\{E'_1, \ldots, E'_s\}$ with adjacency graph $\Gamma(\{E'_1, \ldots, E'_s\})$ isomorphic to Γ and there exists a map $\psi \colon E' \to E$ which sends each cube E'_i isometrically to E_i. We call ψ a *bending of E'*. Sometimes, as in Figure 4, a fold will be indicated by a heavy dashed line.

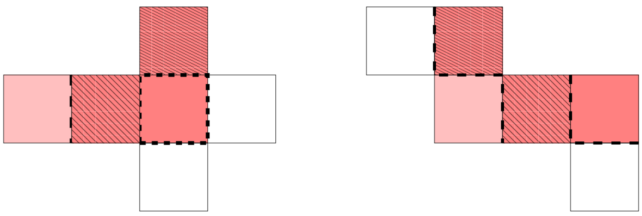

FIGURE 4. Two fold-outs of faces of a 3-cube along non-isomorphic maximal trees.

Fold-out figures, in particular, illustrate 3-cells contained in 3-cubes. Most of our figures of this type, e.g. in Sections 4 and 5, are akin to the following two simple examples.

Consider the cube $Q = [0,3]^3$. Then $F = [0,3]^2 \times \{0\}$ is a face of Q and the unit cube $q = [1,2]^2 \times [0,1]$ is contained in Q and meets F in the face $f = [1,2]^2 \times \{0\}$. We illustrate the fact that q meets F by identifying f in F as in Figure 5.

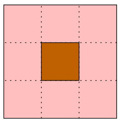

Figure 5. Cube q in Q realized as a square f in F.

In our second example, Q and F remain the cube $[0,3]^3$ and its face $[0,3]^2 \times \{0\}$ respectively, but $q = [0,1] \times [1,2] \times [0,1]$. Let also F' be the face $\{0\} \times [0,3]^2$ of Q. Then $q \cap (F \cup F')$ is a union of two faces $f = [0,1] \times [1,2] \times \{0\}$ and $f' = \{0\} \times [1,2] \times [0,1]$ of q. To indicate how q meets $F \cup F'$ in more than one face, we indicate multiple counting with the symbol 'x', as for f' in Figure 6.

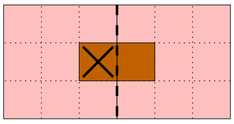

Figure 6. Cube q in Q meeting faces $F \cup F'$.

3. Atoms and molecules

In this section we discuss the elementary BLD-theory of certain cubical n-cells. We call these classes of cells *atoms*, *molecules*, *dented atoms* and *dented molecules*.

Definition 3.1. *We say that a cubical n-cell $A = |P|$ in $\mathbb{R}^n$ is an* atom of length *ℓ if A is r-fine and the adjacency graph $\Gamma(P)$ is a tree of length ℓ.*

Given an atom $A = |P|$, we denote by $\ell(A)$ its length; i.e. $\ell(A) = \ell(\Gamma(P))$. Note also that every r-fine atom A has uniquely determined r-fine complex P_A with $A = |P_A|$.

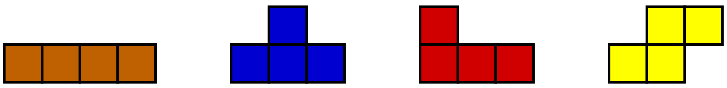

Figure 7. Some atoms of length 4.

Clearly, by finiteness of adjacency trees, every r-fine atom of length ℓ is uniformly L-bilipschitz to the n-cube $[0,r]^n$ with L depending only on n and ℓ. In what follows, we define more complicated cells, using atoms as building blocks. The hierarchy between atoms in these constructions is given by the notion of proper adjacency. Atoms $A = |P|$ and $A' = |P'|$ are *properly h-adjacent*, for $h > 1$, if

(1) $\rho(A) \geq h\rho(A')$ or $\rho(A') \geq h\rho(A)$, and
(2) there exist n-cubes $Q \in P^{(n)}$ and $Q' \in (P')^{(n)}$ for which $A \cap A' = Q \cap Q'$.

Let $\mathcal{A}$ be a finite collection of properly adjacent atoms so that $\Gamma(\mathcal{A})$ is a tree. Suppose also that $\Gamma(\mathcal{A})$ is *John*, that is, the function $A \mapsto \rho(A)$ is a John function on $\Gamma(\mathcal{A})$. Then there exists a unique vertex $\hat{A} \in \mathcal{A}$ so that $\rho(\hat{A}) = \max_{A \in \mathcal{A}} \rho(A)$.

Let $A \in \Gamma(\mathcal{A})$ be an inner vertex in $(\Gamma(\mathcal{A}), \hat{A})$ and let $\mathcal{N}(A)$ be neighbors of A in $\Gamma(\mathcal{A})$. Since $\Gamma(\mathcal{A})$ is John, there exists a unique $A' \in \mathcal{N}(A)$ so that $\rho(A') > \rho(A)$. Let $F_{A'}$ be the face of a cube $Q' \in \Gamma(P_{A'}^{(n)})$ having $A' \cap A \subset F_{A'}$. Similarly, for every $a \in \mathcal{N}(A) \setminus \{A'\}$, let F_a be the face of a cube $q_a \in \Gamma(P_a^{(n)})$ containing $a \cap A$.

Definition 3.2. *An inner vertex A in $(\Gamma(\mathcal{A}), \hat{A})$ is* λ-collapsible *for $\lambda > 1$ if there exists a collection $\{F'_a \subset F_{A'} : a \in \mathcal{N}(A) \setminus \{a\}\}$ of essentially pair-wise disjoint $(n-1)$-cubes with $\rho(F'_a) = \lambda\rho(F_a)$; see Proposition 3.5 for the heuristics behind this terminology.*

Definition 3.3. *Let $M = |\Gamma(\mathcal{A})| = \bigcup_{A\in\mathcal{A}} A$ be a cubical n-cell having an essential partition into finite collection $\mathcal{A}$ of atoms, $\nu \geq 1$ and $\lambda > 1$. Then M is a (ν, λ)-* molecule *if*

(a) *the adjacency graph $\Gamma(\mathcal{A})$ is a tree,*
(b) *adjacent atoms in $\mathcal{A}$ are properly 3-adjacent,*
(c) *$\Gamma(\mathcal{A})$ is John,*
(d) *$\Gamma(\mathcal{A})$ has valence at most ν, and*
(e) *each inner vertex of $(\Gamma(\mathcal{A}), \hat{A})$ is λ-collapsible, where $\hat{A} \in \Gamma(\mathcal{A})$ is the unique atom of largest side length.*

Remark 3.4. *By (c), $M = |\Gamma(\mathcal{A})|$ is a John domain; see e.g.* [8] *or* [19] *for terminology.*

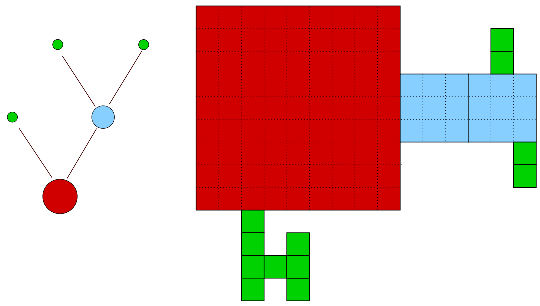

FIGURE 8. Example of a tree $\Gamma(\mathcal{A})$ and molecule $M = |\Gamma(\mathcal{A})|$.

Let $M = |\Gamma(\mathcal{A})|$ be a molecule. By (b), the atoms in $\mathcal{A}$ and the tree $\Gamma(\mathcal{A})$ are uniquely determined. We call $\Gamma(M) = \Gamma(\mathcal{A})$ the *atom tree of M*. We sometimes use the more detailed *internal tree $\Gamma^{\mathrm{int}}(M)$ of M*, where $\Gamma^{\mathrm{int}}(M) = \Gamma\left(\bigcup_{A\in\mathcal{A}} P_A^{(n)}\right)$. The atom $\hat{A}$ in (e) is the *root of M*.

In addition, we say that

$$\ell_{\mathrm{atom}}(M) = \max_{A\in\Gamma(\mathcal{A})} \ell(A)$$

is the *atom length of M*, and that

$$\ell(M) = \ell(\Gamma(\mathcal{A}))$$

is the *(external) length of M*. The *(maximal) side length of M* is

$$\rho(M) = \max_{A\in\Gamma(\mathcal{A})} \rho(A).$$

The main result on molecules is the following bilipschitz contraction property.

Proposition 3.5. *Let M be a (ν, λ)-molecule with root $\hat{A}$ in $\mathbb{R}^n$. Then there exists an L-bilipschitz homeomorphism*

$$\phi\colon\ (M, d_M) \to (\hat{A}, d_{\hat{A}})$$

which is identity on $\hat{A} \cap \partial M$, where L depends only on n, ν, λ, and $\ell_{\mathrm{atom}}(M)$.

This proposition should not surprise any expert. Its proof is based on the bounded local structure of $\Gamma(M)$ and bilipschitz equivalence of atoms of uniformly bounded length. Due to the specific nature of the statement and its fundamental rôle in our arguments, we discuss its proof in detail. We gratefully acknowledge work of Semmes, especially [17], as the main source of these ideas.

The proof of Proposition 3.5 is by induction on the size of the tree $\Gamma(M)$. We begin with a lemma corresponding the induction step of this proof. Given sets X and Y in $\mathbb{R}^n$, the set

$$X \star Y = \{tx + (1-t)y \in \mathbb{R}^n : x \in X,\ y \in Y,\ t \in [0,1]\},$$

is the *join of X and Y*. If Q is an n-cube in $\mathbb{R}^n$, x_Q is its barycenter, that is, $Q = B_\infty(x_Q, r_Q)$ where $r_Q > 0$. For $(n-1)$-cubes F, the barycenter x_F is defined as the average of the vertices of F. Both definitions coincide for n-cubes.

Lemma 3.6. *Let Q be an n-cube and let M be a molecule properly adjacent to Q, $\nu \geq 1$ and $\lambda > 1$. Let F be the face of Q containing $M \cap Q$.*

Let $F_1, \ldots, F_\nu \subset \partial M - Q$ be pair-wise disjoint faces of n-cubes $Q_1, \ldots, Q_\nu$ in $\Gamma^{\mathrm{int}}(M)$, respectively. Suppose there exist essentially pair-wise disjoint $(n-1)$-cubes $F'_1, \ldots, F'_\nu$ in F satisfying $\rho(F'_i) = \lambda\rho(F_i)$ for every $i = 1, \ldots, \nu$.

Then there exist $L = L(n, \ell_{\mathrm{atom}}(M), \ell(M), \nu, \lambda) \geq 1$ and an L-bilipschitz homeomorphism

$$\phi \colon (M \cup Q, d_{M\cup Q}) \to Q,$$

which is the identity on $Q - (F \star \{x_Q\})$ and an isometry on each $F_i \star \{x_{Q_i}\}$.

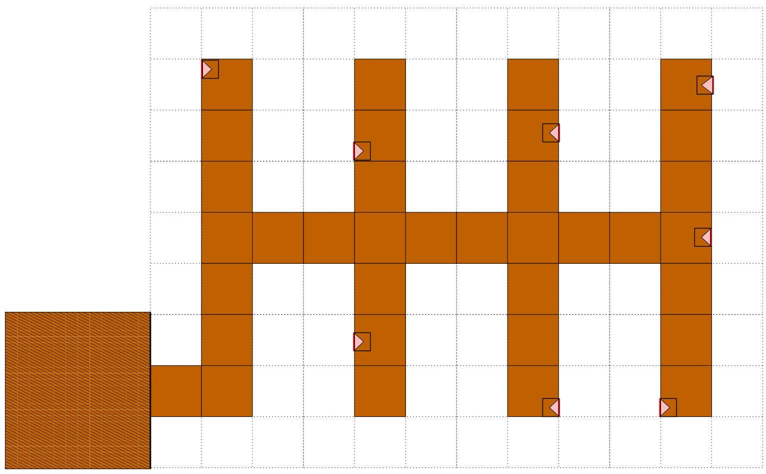

Figure 9. Cube Q, molecule M, and cones $F_i \star \{x_{Q_i}\}$ in cubes Q_i in Lemma 3.6.

Proof. Let $i \in \{1, \ldots, \nu\}$. Let $F''_i = B_\infty(x_{F'_i}, \rho(F_i)/2) \cap F$. Then F''_i is an $(n-1)$-cube in F with the same barycenter as F'_i and the same side length as F_i. We denote by $Q''_i \subset Q$ the n-cube having F''_i as a face, and set $\Delta_i = F_i \star \{x_{Q_i}\}$, $\Delta''_i = F''_i \star \{x_{Q''_i}\}$.

By a shelling argument, there exists a PL-homeomorphism $\phi \colon M \cup Q \to Q$ which is identity outside $F \star \{x_Q\}$ and restricts to an isometry $\phi|\Delta_i \colon \Delta_i \to \Delta''_i$ for every $i = 1, \ldots, \nu$; see e.g. [16, Lemma 3.25]. Since it suffices to consider only a finite number of triangulations and PL-homeomorphisms, ϕ is uniformly bilipschitz with a constant depending only on n, $\ell_{\mathrm{atom}}(M)$, $\ell(M)$, ν, and λ. $\square$

Proof of Proposition 3.5. Let $M = |\Gamma(\mathcal{A})|$ be a (ν,λ)-molecule with root $\hat{A}$. We may assume that $M \neq \hat{A}$ and, more precisely, that $\Gamma(\mathcal{A})$ has inner vertices, since otherwise the claim follows from Lemma 3.6.

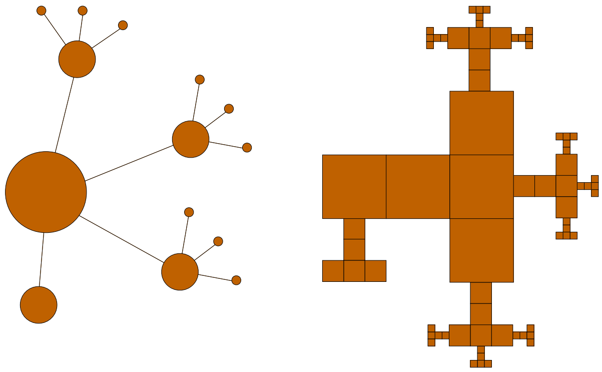

FIGURE 10. A tree $\Gamma(\mathcal{A})$ and molecule $M = |\Gamma(\mathcal{A})|$.

To begin the induction, denote $\Gamma_0 = \Gamma(\mathcal{A})$, $M_0 = M$, and to each leaf $L \in \Gamma_0$ associate a face F_L of an n-cube $Q_L \in \Gamma^{\mathrm{int}}(L)$ with $F_L \subset \partial M_0 \cap L$. We denote the set of these chosen faces by $\mathcal{F}_0$, and for every leaf $L \in \Gamma_0$ set $J_L = F_L \star \{x_{Q_L}\}$.

Fix an atom $A_0' \in \Gamma_0$ which is an inner vertex in Γ_0 so that the rooted subtree $\Gamma_0' = \Gamma_{A_0'}$ behind A_0' in $(\Gamma_0, \hat{A})$ consists of leaves of Γ_0. Also choose an atom $A_0 \in \Gamma_0 \setminus \Gamma_0'$ adjacent to A_0' in Γ_0. Let Q_0 be the unique n-cube in A_0 and F_0 the unique face of Q_0 which contains $A_0 \cap A_0'$; denote $J_0 = F_0 \star \{x_{Q_0}\}$ and $\mathcal{F}_0' = \{F_L : L \in \Gamma_0'\}$.

Since $M = |\Gamma(\mathcal{A})|$ is a (ν,λ)-molecule and A_0' is an inner vertex in $\Gamma(\mathcal{A})$, A_0' is λ-collapsible. Thus there exists a collection $\{F_L' : L \in \Gamma_0'\}$ of pair-wise disjoint $(n-1)$-cubes satisfying $\rho(F_L') = \lambda\rho(F_L)$ for every $L \in \Gamma_0'$.

By Lemma 3.6, there exist a constant $L \geq 1$, depending only on n, ν, δ, $\ell_{\mathrm{atom}}(M)$, and $\ell(M)$, and an L-bilipschitz homeomorphism

$$\phi_0 : (|\Gamma_0'| \cup Q_0, d_{|\Gamma_0'| \cup Q_0}) \to (Q_0, d_{Q_0}),$$

which is the identity on $Q - (F_0 \star \{x_{Q_0}\})$ and an isometry on the join J_L for $L \in \Gamma_0'$.

We now define $\Gamma_1 = \Gamma_0 \setminus \Gamma_0'$ and $\mathcal{F}_1 = (\mathcal{F}_0 \setminus \mathcal{F}_0') \cup \{F_0\}$. Then $M_1 = |\Gamma_1|$ is a (ν,λ)-molecule with root $\hat{A}$. In terms of this notation, ϕ_0 extends, by identity, to an L-bilipschitz homeomorphism

$$\phi_0 : (M_0, d_{M_0}) \to (M_1, d_{M_1}),$$

which is an isometry on every join $J_L = F_L \star \{x_{Q_L}\}$ for $L \in \Gamma_0'$.

Clearly, $\ell(M_1) < \ell(M_0)$. We iterate this step to obtain a descending sequence of subgraphs $\Gamma_0, \ldots, \Gamma_i$ of $\Gamma(\mathcal{A})$ so that every Γ_j has at least one vertex fewer than Γ_{j-1} for $j = 1, \ldots, i$. Since $\Gamma(\mathcal{A})$ is a finite tree, there exists $i_0 \geq 1$ depending on $r(\Gamma(\mathcal{A}), \hat{A})$ so that Γ_{i_0} consists of only $\hat{A}$.

For $i = 0, \ldots, i_0$, we also obtain collections of faces $\mathcal{F}_0, \ldots, \mathcal{F}_i$ on leaves of graphs $\Gamma_0, \ldots, \Gamma_i$, and L-bilipschitz homeomorphisms

$$\phi_{j-1} : (|\Gamma_{j-1}|, d_{|\Gamma_{j-1}|}) \to (|\Gamma_j|, d_{|\Gamma_j|})$$

which are isometries on the joins over the faces in $\mathcal{F}_{j-1}$ for every $j = 1, \ldots, i_0$. As in the construction above, $\phi_i(|\Gamma_{i-1}|)$ is contained in a join over a face in $\mathcal{F}_i$. Thus

$$\phi_i \circ \cdots \circ \phi_0 : (|\Gamma_0|, d_{|\Gamma_0|}) \to (|\Gamma_i|, d_{|\Gamma_i|})$$

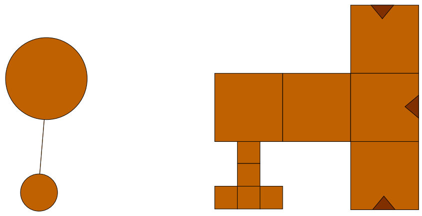

FIGURE 11. An intermediate tree Γ_i and cell $|\Gamma_i|$.

is L-bilipschitz for every $i = 0, \ldots, i_0$, where L depends only on n, ν, λ, and $\ell_{\mathrm{atom}}(M)$, and so

$$\phi_{i_0} \circ \cdots \circ \phi_0 \colon (|\Gamma_0|, d_{|\Gamma_0|}) \to (\hat{A}, d_{\hat{A}})$$

satisfies the conditions of the claim. This concludes the proof. □

Corollary 3.7. *Let $M = |\Gamma(\mathcal{A})|$ be a (ν, λ)-molecule and let $\Gamma \subset \Gamma(\mathcal{A})$ be a subtree containing the root $\hat{A}$ of M. Then there exist an $L \geq 1$ depending only on n, ν, λ, and $\ell_{\mathrm{atom}}(M)$, and an L-bilipschitz homeomorphism $\phi \colon (M, d_M) \to (|\Gamma|, d_{|\Gamma|})$ which is the identity on $|\Gamma| \cap \partial M$.*

Proof. Let Γ' be a component of $\Gamma(\mathcal{A}) \setminus \Gamma$. Then $|\Gamma|$ is an (ν, λ)-molecule. Thus the claim follows by applying Proposition 3.5 to components of $\Gamma(\mathcal{A}) \setminus \Gamma$ followed by Lemma 3.6 on the roots of these trees. □

Before introducing dented atoms, we record a uniform bilipschitz equivalence result in spirit of Proposition 3.5, and leave the details to the interested reader.

Proposition 3.8. *Let $\nu \geq 1$, $\lambda > 1$, $\ell \geq 1$, and let (M_m) be an increasing sequence of (ν, λ)-molecules so that, for every $m \geq 1$,*

1. *$M_m - M_{m-1}$ is connected and contains the root of M_m,*
2. *$\ell_{\mathrm{atom}}(M_m) \leq \ell$, and*
3. *if A and A' are adjacent in $\Gamma(M_m)$ with $\rho(A) < \rho(A')$ then $\rho(A') = 3\rho(A)$.*

Let $M = \bigcup_{m \geq 0} M_m$. Then (M, d_M) is L-bilipschitz equivalent to $\mathbb{R}^{n-1} \times [0, \infty)$, where L depends only on n, ν, λ, and ℓ.

Sketch of proof. Let Γ be the tree $\bigcup_{m \geq 0} \Gamma(M_m)$, and let Γ' be the unique branch passing through all roots $\hat{M}_m$ of M_m for $m \geq 0$. We may consider Γ' as a sequence of atoms with increasing side length, and for every $m \geq 0$ denote by Γ'_m the part of Γ' contained in $\Gamma(M_m)$.

Following the idea of Corollary 3.7, we obtain a sequence (ψ_m) of L'-bilipschitz contractions $\psi_m \colon (M_m, d_{M_m}) \to (|\Gamma'_m|, d_{|\Gamma'_m|})$ so that $\psi_{m+1}|M_m = \psi_m$ for every $m \geq 0$, where L' depends only on n, ν, λ, and ℓ. This produces an L-bilipschitz map $\psi \colon (M, d_M) \to (|\Gamma'|, d_{|\Gamma'|})$.

It remains now to show that $(|\Gamma'|, d_{|\Gamma'|})$ is L''-bilipschitz equivalent to $\mathbb{R}^{n-1} \times [0, \infty)$, where L'' depends only on n and ℓ.

Let A be the unique vertex in Γ' with valence 1. Since Γ' is a branch, we may now enumerate the vertices in Γ' as $A = a_0, a_1, a_2, \ldots$ with a_k adjacent to a_{k+1}. By (3), $\rho(a_{k+1}) = 3\rho(a_k)$ for every $k \geq 0$. Thus $(|\Gamma'|, d_{|\Gamma'|})$ is L'''-bilipschitz equivalent, $L''' = L'''(n, \ell)$, to a cone

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_n^2 \leq x_1^2 + \cdots x_{n-1}^2\},$$

and hence L''-bilipschitz equivalent to $\mathbb{R}^{n-1} \times [0, \infty)$, where L'' depends only on n and ℓ. □

3.1. Dented atoms.

Definition 3.9. *Let A be an atom in $\mathbb{R}^n$. A molecule M contained in A is* on the boundary of A *if $A - M$ an n-cell and for every $Q \in \Gamma^{\mathrm{int}}(M)$*

(i) *Q is contained in a strictly larger cube of $\Gamma^{\mathrm{int}}(M)$, and*
(ii) *$Q \cap \partial A$ contains a face of Q.*

Definition 3.10. *Let A be an atom in $\mathbb{R}^n$ and let $M_1, \ldots, M_\nu \subset A$ be pair-wise disjoint molecules on the boundary of A each having side length at most $3^{-2}\rho(A)$. The n-cell $D = A - \bigcup_i M_i$ is a* dented atom *if*

(i) *each M_i is contained in an n-cube in $\Gamma(A)$, and*
(ii) *$\mathrm{dist}\,(Q, Q') \geq \min\{\rho(Q), \rho(Q')\}$ for all $Q \in \Gamma^{\mathrm{int}}(M_i)$ and $Q' \in \Gamma^{\mathrm{int}}(M_j)$ for $i \neq j$.*

The molecules $M_1, \ldots, M_\nu$ are called dents of A*, and the atom A is the* hull of D*,* hull(D).

Remark 3.11. *The reader may find the constant 3^{-2} curious, but this explicit constant is chosen to be compatible with constructions in Section 5. These constructions also have the property that each cube in $\Gamma(A)$ has at most 2 dents.*

By (ii), the hull and the dents of a dented atom are unique. Given a dented atom $D = A - \bigcup_{i=1}^{\nu} M_i$, we write $\Sigma(D) = \bigcup_i \Gamma(D_i)$, $\Sigma^{\mathrm{int}}(D) = \bigcup_i \Gamma^{\mathrm{int}}(D_i)$, and $\rho(D) = \rho(\mathrm{hull}(D))$. For notational consistency, we consider every atom as a (trivially) dented atom and define $\mathrm{hull}(A) = A$ for every molecule A. When $\mathrm{hull}(D)$ is a cube, D is a *dented cube*.

The main result on dented atoms is the following uniform bilipschitz restoration result. We note that neither the internal geometry of the hull nor the geometry of dents have a rôle in the statement. This is a consequence of confining the dents to be in cubes of the hull and the local nature of the construction of the homeomorphism.

Proposition 3.12. *Suppose D is a dented atom with hull A. Then there exists $L = L(n)$ and an L-bilipschitz homeomorphism $\phi\colon (D, d_D) \to (A, d_A)$ which is the identity on $D \cap \partial A$.*

We begin the proof by a simple observation on neighborhoods of cubes. Let Q and $q = B_\infty(x_q, r_q)$ be n-cubes in $\mathbb{R}^n$ so that $q \subset Q$ and q has a face in ∂Q. The set

$$\mathrm{Cone}(q, Q) = \{x \in B_\infty(x_q, (7/6)r_q) \cap Q \colon 2\mathrm{dist}\,(x, q) \leq \mathrm{dist}\,(x, \partial Q)\}$$

is the *truncated conical neighborhood of q in Q*.

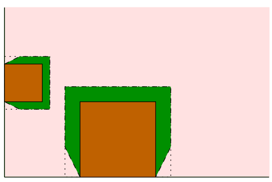

FIGURE 12. Two cubes and their (truncated) conical neighborhoods in a larger cube.

Lemma 3.13. *Let $D = A - \bigcup_i D_i$ be a dented atom in $\mathbb{R}^n$. Then there exists $\mu > 0$ depending only on n so that*

$$\#\{q' \in \Sigma(D) \colon \mathrm{Cone}(q', Q) \cap \mathrm{Cone}(q, Q) \neq \varnothing\} \leq \mu$$

for all $q \in \Sigma(D)$.

Proof. Let q and q' be n-cubes in an n-cube Q so that q and q' have a face in ∂Q. Suppose also that either $\rho(q) = \rho(q')$ or $\rho(q) \geq 3\rho(q')$, and $\mathrm{dist}_\infty(q, q') \geq \rho(q')$. Then $\mathrm{Cone}(q, Q) \cap \mathrm{Cone}(q', Q) = \varnothing$.

Suppose now that we have a dented atom $D = A - \bigcup_i D_i$, and n-cubes q and q' in $\Sigma(D)$ contained in $Q \in \Gamma(A)$. Then, by definition of dented atom and the first observation, $\mathrm{Cone}(q, Q) \cap \mathrm{Cone}(q', Q) \neq \varnothing$ if and only if $q \cap q' \neq \varnothing$. Thus truncated conical neighborhoods of q and q' meet if and only if q and q' meet. Hence it suffices that μ be larger than the number of neighbors of q of the same side length, so we take $\mu = 3^n$. $\square$

Remark 3.14. *Let D be a dented atom and consider cubes $Q, Q' \in \Gamma(\mathrm{hull}(D))$, $Q \neq Q'$. Then* $\mathrm{Cone}(q, Q) \cap \mathrm{Cone}(q', Q') = \varnothing$ *for $q, q' \in \Sigma(D)$ if $q \subset Q$ and $q' \subset Q'$.*

Proof of Proposition 3.12. The proof is an inductive collapsing of $A - D$ along the forest $\Sigma(D)$ removing leaves one by one. Let $m = \#\Sigma(D)$.

Let Σ be a subforest of $\Sigma(D)$, $q \in \Sigma$ a leaf, $Q \in \Gamma(A)$ be the cube containing q, and denote $\Sigma' = \Sigma \setminus \{q\}$. Then there exists a PL homeomorphism $\phi_{\Sigma,q} \colon A - |\Sigma| \to A - |\Sigma'|$ having support in $\mathrm{Cone}(q, Q)$; that is $\phi_{\Sigma,q}(x) = x$ for $x \notin \mathrm{Cone}(q, Q)$. Clearly, we may take $\phi_{\Sigma,q}$ L-bilipschitz with L depending only on n.

With this observation at our disposal, we find a sequence $\Sigma(D) = \Sigma_0 \supset \cdots \supset \Sigma_m = \varnothing$ of forests and L-bilipschitz PL-homeomorphisms $\phi_i \colon A - |\Sigma_{i-1}| \to A - |\Sigma_i|$ having support in the conical neighborhood of the leaf $\Sigma_{i-1} \setminus \Sigma_i$ for every $i = 1, \ldots, m$.

Since the number of cones over cubes in Σ is locally bounded by Lemma 3.13,

$$\phi = \phi_m \circ \cdots \circ \phi_0 \colon (D, d_D) \to (A, d_A)$$

is a bilipschitz homeomorphism with a bilipschitz constant depending only on n. $\square$

3.2. **Dented molecules.** We end this section by defining dented molecules, which relate to dented atoms as molecules relate to atoms.

Definition 3.15. *A dented atom D' is* properly adjacent to a dented atom D *if* $\mathrm{hull}(D') \cup \mathrm{hull}(D)$ *is a molecule and either*

1. $\mathrm{hull}(D') \subset \mathrm{hull}(D)$ *and* $D' \cap D = \mathrm{hull}(D') \cap D$, *or*
2. $\mathrm{hull}(D') \cap \mathrm{hull}(D) = D' \cap D$.

Note that, by (1), proper adjacency is not a symmetric relation. However, we symmetrize this relation by saying that dented atoms D and D' are *properly adjacent* if D' is properly adjacent to D or D is properly adjacent to D'.

Let $\mathcal{D}$ be a finite collection of dented atoms so that each pair of atoms in $\mathcal{D}$ is either properly adjacent or pair-wise disjoint. Since dented atoms are n-cells, the adjacency tree $\Gamma(\mathcal{D})$ is well-defined. Let $U = |\Gamma(\mathcal{D})|$ and let $M = \bigcup_{D \in \mathcal{D}} \mathrm{hull}(D)$. By proper adjacency of the dented atoms, M is a molecule.

Definition 3.16. *An n-cell U is a* dented molecule *if there exists a finite collection $\mathcal{D}$ of pair-wise properly adjacent dented atoms so that $\Gamma(\mathcal{D})$ is a tree and $U = |\Gamma(\mathcal{D})|$. The n-cell* $\mathrm{hull}(U) = \bigcup_{D \in \mathcal{D}} \mathrm{hull}(D)$ *is the* hull of U. *The vertex $\hat{D} \in \mathcal{D}$ is the* root of U *if* $\mathrm{hull}(\hat{D})$ *is the root of* $\mathrm{hull}(U)$.

Remark 3.17. *Note that, given a dented molecule $U = |\Gamma(\mathcal{D})|$, the collection $\mathcal{D}$ is uniquely determined. We call elements of $\mathcal{D}$ the* dented atoms of U *and define* $\Gamma(U) = \Gamma(\mathcal{D})$.

Let U be a dented molecule. We define internal and external vertices of $\Gamma(U)$ as follows.

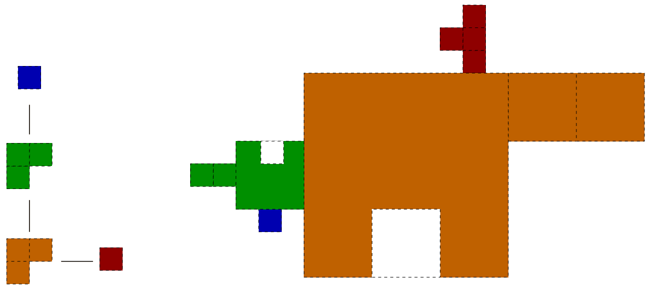

FIGURE 13. A dented molecule U with a tree $\Gamma(U)$.

Definition 3.18. *A dented atom $D \in \Gamma(U)$ is* internal *if there exists $D' \in \Gamma(U)$ satisfying $D \subset \mathrm{hull}(D')$. A dented atom in $\Gamma(U)$ is* external *if it is not internal. We denote by $\Gamma_I(U)$ the set of internal vertices of $\Gamma(U)$ and by $\Gamma_E(U)$ the set of external vertices.*

The motivation for this dichotomy is the following easy observation, which we record as a lemma.

Lemma 3.19. *Let U be a dented molecule. Then $D \mapsto \mathrm{hull}(D)$ is a tree isomorphism $\Gamma_E(U) \to \Gamma(\mathrm{hull}(U))$. In particular,*

$$\mathrm{hull}(U) = \bigcup_{D \in \Gamma_E(U)} \mathrm{hull}(D).$$

We finish this section by introducing terminology related to dented molecules. Let D be a dented molecule.

Definition 3.20. *A vertex $d \in \Gamma(D)$ is* expanding in D *if the subtree $\Gamma(D)_d$ behind d in $\Gamma(D)$ consists of atoms.*

Note that, if d is expanding in D then d is an atom, since $d \in \Gamma(D)_d$.

Definition 3.21. *Given $d \in \Gamma(D)$, a vertex $d' \in \Gamma(D)$ is a* parent of d *if d' is adjacent to d, $\rho(d') > \rho(d)$, and $d \subset \mathrm{hull}(d')$; in this case, d is a* child of d'.

Note that a vertex need not have a parent.

Definition 3.22. *A dented molecule D' is a* partial hull of D *if there exist vertices $d_1, \ldots, d_\ell$ of $\Gamma(D)$ for which*

$$D' = D \cup \bigcup_{k=1}^{\ell} \mathrm{hull}(d_k).$$

Remark 3.23. *In Section 5, we consider a sequence of dented molecules (U_i) for which $\mathrm{hull}(U_i)$ is a (ν, λ)-molecule with ν and λ depending only on n, but the adjacency tree $\Gamma(U_i)$ does not have a uniformly bounded valence.*

However, the structure property of Lemma 3.19 can, in the case of this sequence (U_i), be realized on the level of cells. More precisely, we will show there exist L-bilipschitz maps $U_i \to \mathrm{hull}(U_i)$ with L depending only on n. This proof is based on a sequence of partial hulls from U_i to $\mathrm{hull}(U_i)$.

Since we prove this statement only for particular dented molecules based on notions in the following section, we postpone this statement to Section 5. Nevertheless, we invite the interested reader to consider a general statement along the lines of Propositions 3.5 and 3.12.

4. Local rearrangements and the tripod property

In this section we develop tools to produce rough Rickman partitions. Throughout this section we consider different kinds of repartitions in a single cube. These repartitions are related to the final essential partitions introduced in Section 5 only tangentially, so the reader may find these constructions unmotivated. Our aim is to simplify these later discussions by introducing these local modifications and their properties here before exploiting them later. Thus the reader should consider the whole section as a preparation for Section 5.

To motivate the rôle of our tools, consider the following example. Let D_1, D_2, and D_3 be the cubes $[0,1]^{n-1}\times[0,1]$, $[0,1]^{n-1}\times[-1,0]$, and $[1,2]\times[0,1]^{n-2}\times[0,1]$, respectively, $\mathbf{D}$ the essential partition (D_1,D_2,D_3) of their union.

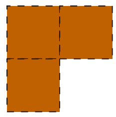

Figure 14. Essential partition $\mathbf{D}$.

The Hausdorff distance of the common boundary $\partial_\cap \mathbf{D}$ and the pair-wise common boundary $\partial_\cup \mathbf{D}$ satisfy

$$\operatorname{dist}_{\mathcal{H}}(\partial_\cup \mathbf{D}, \partial_\cap \mathbf{D}) = 1 \tag{4.1}$$

in the sup-metric.

Let $k>0$ and consider now the sets $V_i = 3^k D_i$ for $i=1,2,3$, and associated essential partition $\mathbf{V}=(V_1,V_2,V_3)$. Of course, topological properties and bilipschitz equivalence of the cubes remain invariant under this scaling. The Hausdorff-distances in (4.1) scale accordingly, and so

$$\operatorname{dist}_{\mathcal{H}}(\partial_\cup \mathbf{V}, \partial_\cap \mathbf{V}) = 3^k. \tag{4.2}$$

We will show that in this case, as well as in more general situations, there exists an essential partition $\mathbf{W}=(W_1,W_2,W_3)$ of $\bigcup_i V_i$ into n-cells (W_i, d_{W_i}) uniformly bilipschitz to $[0,3^k]^n$ with

$$\operatorname{dist}_{\mathcal{H}}(\partial_\cup \mathbf{W}, \partial_\cap \mathbf{W}) \le 6 \tag{4.3}$$

in the sup-metric.

Property (4.3) is a consequence of the so-called tripod property, informally mentioned in the introduction, which we now formally define. We first need an equivalence relation. Let U be a 3-fine cubical n-set in $\mathbb{R}^n$ and let U^* be a 3-fine subdivision of U. Suppose $\mathbf{U}=(U_1,U_2,U_3)$ is an essential partition of U, and let $(\partial_\cup \mathbf{U})^\#$ be the unit subdivision of $\partial_\cup \mathbf{U}$ as defined in Section 2.2. Let $\Gamma_\cup(\mathbf{U})$ be the subgraph of the adjacency graph $\Gamma((\partial_\cup \mathbf{U})^\#)$ composed of vertices of $\Gamma((\partial_\cup \mathbf{U})^\#)$ and all edges $\{q,q'\}\in\Gamma((\partial_\cup \mathbf{U})^\#)$ for which $q\cup q'\subset U_i\cap U_j$ for a pair $i\ne j$.

Example 4.1. *In the discussion accompanying Figure 14,* $\Gamma(\mathbf{D})$ *consists of two vertices* $\{[0,1]^{n-1}\times\{0\}, \{1\}\times[0,1]^{n-1}\}$ *and has no edges, whereas* $\Gamma(\mathbf{V})$ *is a forest of two non-trivial trees, since* $k>0$.

Definition 4.2. *Unit* $(n-1)$*-cubes* q *and* q' *in* $(\partial_\cup \mathbf{U})^\#$ *are* $\mathbf{U}$-equivalent *if*

(a) q *and* q' *are in the same component of* $\Gamma_\cup(\mathbf{U})$ *and*
(b) $q\cup q'\subset Q$ *for some* $Q\in U^*$.

Denote by $[q]$ the $\mathbf{U}$-equivalence class of $q\in(\partial_\cup \mathbf{U})^\#$ and by $|[q]|$ the union $\bigcup_{q'\in[q]} q'$. For each pair (i,j), $i\ne j$, the $\mathbf{U}$-equivalence class $[q]$ of $q\in(\partial_\cup \mathbf{U})^\#$ is said to be *between* U_i *and* U_j when $q\subset U_i\cap U_j$.

Remark 4.3. *Condition (b) in Definition 4.2 implies that the equivalence class* $[q]$ *of* $q \in (\partial_\cup \mathbf{U})^\#$ *has diameter at most* 3 *in the sup-metric. Note that equivalence classes are cubical* 1*-fine sets of dimension* $n-1$*, and that the number of* $(n-1)$*-cubes in* $[q]$ *is uniformly bounded by a constant depending only on* n*.*

Definition 4.4. *An essential partition* $\mathbf{U}$ *of* U *has the* tripod property *if there exists an essential partition* Δ *of* $\partial_\cup \mathbf{U}$ *into cubical* $(n-1)$*-cells satisfying*

- ($\Delta 1$) *each* $c \in \Delta$ *is contained in a* $\mathbf{U}$*-equivalence class, and*
- ($\Delta 2$) *to each* $c_1 \in \Delta$ *corresponds a unique pair* $c_2, c_3 \in \Delta$ *for which* $c_1 \cap c_2 \cap c_3$ *is an* $(n-2)$*-cell in* $\partial_\cap \mathbf{U}$ *and* c_1, c_2, c_3 *are contained in different* $\mathbf{U}$*-equivalence classes.*

The tripod property of an essential partition is most conveniently verified using the following local tripod property.

Definition 4.5. *Given an essential partition* $\mathbf{U}$ *and a cube* $Q \subset U$ *of side length at least* 3*, we say that* $\mathbf{U}$ has the tripod property relative to Q *if there exists an essential partition* Δ *of* $Q \cap \partial_\cup \mathbf{U}$ *into* $(n-1)$*-cells satisfying* ($\Delta 1$) *and* ($\Delta 2$)*.*

Example 4.6. *To give a simple example of an essential partition* $\mathbf{U}$ *satisfying the tripod property we consider* $\mathbf{U} = (Q - A, A, Q')$*, where* $Q = [0,3]^3$*,* $Q' = [0,3]^2 \times [-3,0]$*, and* A *the atom* $A = \bigcup_{r=1}^4 q_r$*, where* $q_r = [r-1, r] \times [1,2] \times [0,1]$ *for* $r = 1,2,3$ *and* $q_4 = [1,2] \times [2,3] \times [0,1]$*; see Figure 15.*

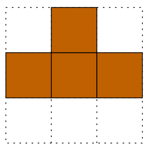

FIGURE 15. Profile of q_1, q_2, q_3, q_4 on the face common to Q and Q'.

Note first that $(Q - A) \cap Q'$ *has three components* $f_1 = [0,1] \times [2,3] \times \{0\}$*,* $f_2 = [0,3] \times [0,1] \times \{0\}$*, and* $f_3 = [2,3] \times [2,3] \times \{0\}$*, whereas* $A \cap (Q - A)$ *and* $A \cap Q'$ *are* 2*-cells. We organize the essential partition* Δ *of* $\partial_\cup \mathbf{U}$ *into three triples* Δ_1*,* Δ_2*, and* Δ_3 *by subdividing cells* $A \cap (Q - A)$ *and* $A \cap Q'$ *as follows.*

For $r = 1, 3$*, we set* $\Delta_r = \{f_r, q_r \cap (Q - A), q_r \cap Q'\}$*. Let* $\Delta_2 = \{f_2, (q_2 \cup q_4) \cap (Q - A), (q_2 \cup q_4) \cap Q'\}$*. For each* r*, we directly check that* Δ_r *is a triple of* $(n-1)$*-cells. In addition,* $\bigcap_{c \in \Delta_r} c$ *is an* $(n-2)$*-cell for every* $r = 1,2,3$*. Hence* $\Delta = \bigcup_{r=1}^3 \Delta_r$ *is an essential partition of* $\partial_\cup \mathbf{U}$ *satisfying conditions* ($\Delta 1$) *and* ($\Delta 2$)*.*

4.1. **Building blocks.** We introduce the elementary atoms which generate rough Rickman partitions.

An $(n-1)$-cell F in $\mathbb{R}^n$ is *planar* if F is congruent to an $(n-1)$-cell in $\mathbb{R}^{n-1}$. Suppose P is an r-fine n-cell and F a planar $(n-1)$-cell. We call P *F-based* if there exists an $(n-1)$-cell F' in $\mathbb{R}^{n-1}$ and a cubical $(n-1)$-cell $P' \subset F'$ so that $P \cup F$ is congruent to $(P' \times [0,r]) \cup F' \subset \mathbb{R}^n$.

Let $T_n = \{0, \pm e_1, \ldots, \pm e_n\}$ and let $\mathcal{T}_n$ be the graph with vertices T_n and edges $\{0, e_i\}$ and $\{0, -e_i\}$ for $i = 1, \ldots, n$.

Definition 4.7. *An atom* A *is an* (n-dimensional) building block *if* $\Gamma(A)$ *is isomorphic to a proper subtree of* $\mathcal{T}_{n-1}$ *having at least two vertices.*

Let B be a building block in $\mathbb{R}^n$. Since $\Gamma(B)$ is a proper subtree of $\mathcal{T}_{n-1}$, we observe that, for all $q \in \Gamma(B)$, the cubical set $q \cap \partial B$ is an $(n-1)$-cell which induces an essential partition to the faces of q. Note also that the adjacency graph

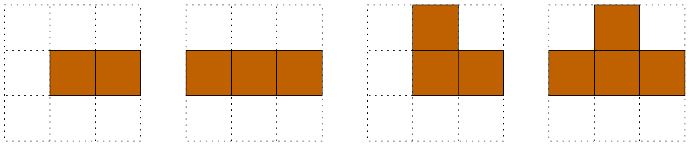

FIGURE 16. Congruence classes of building blocks for $n = 3$.

$\Gamma(q \cap \partial B)$ of these faces is connected. Moreover, $\Gamma(B)$ has valence at most $2(n-1)$ and contains at most one vertex $q \in \Gamma(B)$ having valence greater than 1. More precisely, if $\#\Gamma(B) > 2$ there exits a unique n-cube q_B in B which is a vertex in $\Gamma(B)$ with valence greater than 1: this unique cube q_B is the *center of* B.

A building block B in $\mathbb{R}^n$ is r-*fine* if B is an r-fine atom for some $r > 0$.

Suppose Q is a cube of side length $3r$ containing an r-fine building block B along a face F of Q. Then, for every cube $q \in \Gamma(B)$, $q \cap F$ is an $(n-1)$-cube and a face of q. For the following definition, recall that a *barycenter of a k-cube C* is the unique point in C equidistant from all vertices of C.

Definition 4.8. *Suppose $Q \subset \mathbb{R}^n$ is an n-cube of side length $3r$ containing an F-based r-fine building block $B \subset Q$, where F is a face of Q. Let x_F be the barycenter of F. The building block B is* centered in Q *if either of the following conditions is satisfied:*

1. *if B has a center q_B then x_F is the barycenter of $q_B \cap F$, or*
2. *if $\#\Gamma(B) = 2$, then $\Gamma(B)$ contains a cube q with x_F the barycenter of $q \cap F$.*

The significance of centered building block is motivated by the following observation.

Remark 4.9. *Let $Q \subset \mathbb{R}^n$ be a cube side length 3 and B a 1-fine centered building block contained in Q along the face F of Q. Since B is centered, the barycenter x_F of F is the barycenter of one of the cubes in $\Gamma(B)$, say q_0. Suppose that $q \in \Gamma(B)$ is a cube adjacent to q_0. Since Q has side length 3 and the barycenter of q_0 is x_F, we have that $q \cap (\partial Q - F)$ is a face of q. In particular, the components of $B \cap (\partial Q - F)$ are unit $(n-1)$-cubes, which are in one to one correspondence with cubes in $B - q_0$, cf. Figure 16.*

Convention. *Unless otherwise specified, we assume from now on that every r-fine building block B in a cube Q is centered and based on a face of Q whenever Q has side length $3r$. We extend the notion of* center, *by defining that the unique cube in B containing the barycenter of F on its boundary the* center of B.

Building blocks give rise to a local tripod property of the following form.

Proposition 4.10. *Let $n \geq 3$, and let Q and Q' be n-cubes of side length 3 with a common face $F = Q \cap Q'$, and let B be an F-based building block in Q. Then $\mathbf{U} = (Q - B, B, Q')$ has the tripod property.*

We begin the proof of Proposition 4.10 with a partition lemma.

Lemma 4.11. *Let $n \geq 2$, and let A be a 1-fine atom in $Q = [0,3]^n$ containing the cube $[1,2]^n$ and having $\Gamma(A)$ isomorphic to a subgraph of $\mathcal{T}_n$ satisfying $1 < \#\Gamma(A) \leq 2n$. Then $Q - A$ has an essential partition $\mathcal{P}$ into n-cells. Moreover, there exist cubes $\mathcal{C}_{\mathcal{P}} = \{q_C \in A^\# \colon C \in \mathcal{P}\}$ so that $q_C \neq q_{C'}$ for cells $C \neq C'$ in $\mathcal{P}$ and $q_C \cap C$ contains an $(n-1)$-cube for every $C \in \mathcal{P}$.*

Proof. In the special case $\#\Gamma(A) = 2$, we may take $\mathcal{P} = \{Q - A\}$ and $\mathcal{C}_{\mathcal{P}} = \{[1,2]^n\}$.

The proof in the general case is by induction on the dimension n. The claim clearly holds for $n = 2$; consider e.g variations of Example 4.6. Suppose that $n \geq 3$ is a dimension for which the claim holds for $n - 1$.

Let A be a 1-fine atom in $Q = [0,3]^n$ containing $[1,2]^n$ with $\Gamma(A)$ isomorphic to a subtree of $\mathcal{T}_n$ and $1 < \#\Gamma(A) \leq 2n$. By rotation, we may assume that $[1,2]^n + e_1 \in \Gamma(A)$. Let $F = [0,3]^{n-1}$. Then $A \cap (F \times [1,2]) = A' \times [1,2]$, where A' is an $(n-1)$-dimensional atom in F where $\Gamma(A')$ is isomorphic to a subgraph of $\mathcal{T}_{n-1}$ and $1 < \#\Gamma(A') \leq 2(n-1)$. By induction, $F - A'$ has an essential partition $\mathcal{P}'$ into $(n-1)$-cells. Thus there is a one-one correspondence $C' \leftrightarrow q_{C'}$, where $C' \in \mathcal{P}'$ and $q_{C'} \in \mathcal{C}_{\mathcal{P}'} \subset A^{\#}$ with $C' \cap q_{C'}$ containing an $(n-2)$-cube.

Let $\mathcal{P}'' = \{C' \times [0,3] \colon C' \in \mathcal{P}'\}$. We observe that $Q - (|\mathcal{P}''| \cup A)$ consists of unit cubes in $(A' \times [0,3] - A)^{\#}$. It is now easy to find, for each $C' \in \mathcal{P}'$ a cubical n-cell $\Omega_{C'}$ so that $C' \times [0,3] \subset \Omega_{C'}$, $\bigcup_{C' \in \mathcal{P}'} \Omega_{C'} = Q - A$, and that the sets $\Omega_{C'}$ are pair-wise essentially disjoint. We set $\mathcal{P} = \{\Omega_{C'} \colon C' \in \mathcal{P}'\}$ and $\mathcal{C}_{\mathcal{P}} = \{q_{C'} \times [1,2] \colon C' \in \mathcal{P}'\}$. $\square$

The following corollary encapsulates the key consequence of Lemma 4.11.

Corollary 4.12. *Let $n \geq 3$, Q an n-cube of side length 3 and F a face of Q. Given an F-based building block B in Q, the set $F - B$ has an essential partition $\mathcal{P}$ into cubical $(n-1)$-cells and there exists a collection $\mathcal{C}_{\mathcal{P}} = \{q_C \in B^{\#} \colon C \in \mathcal{P}\}$ of pair-wise essentially disjoint unit n-cubes so that $C \cap q_C$ contains an $(n-2)$-cube for every $C \in \mathcal{P}$.*

Proof. We may assume $Q = [0,3]^n$ and $F = [0,3]^{n-1}$. Since $F \cap B$ is an $(n-1)$-dimensional atom containing $[1,2]^{n-1}$ and having an adjacency tree isomorphic to a (proper) subtree of $\mathcal{T}_{n-1}$ with at least two vertices, the claim follows from Lemma 4.11. $\square$

Proof of Proposition 4.10. Clearly $\partial_\cup \mathbf{U}$ consists of $\mathbf{U}$-equivalence classes $(Q-B) \cap B$, $B \cap Q'$, and $(Q-B) \cap Q'$. The classes $(Q-B) \cap B$ and $B \cap Q'$ are $(n-1)$-cells meeting $\partial_\cap \mathbf{U}$ in an $(n-2)$-cell. We construct now an essential partition of $\partial_\cup \mathbf{U}$ into $(n-1)$-cells as required.

Let $\mathcal{P}$ and $\mathcal{C}_{\mathcal{P}}$ be as in Corollary 4.12. Then there exist atoms $A_C \subset B$ for $C \in \mathcal{P}$, so that $\{A_C \colon C \in \mathcal{P}\}$ is an essential partition of B and $q_C \subset A_C$ for every $C \in \mathcal{P}$. If for every $C \in \mathcal{P}$, we take $\Delta_C = \{A_C \cap (Q - B), A_C \cap Q', C\}$, then $\Delta = \bigcup_{C \in \mathcal{P}} \Delta_C$ is the required partition of $\partial_\cup \mathbf{U}$. $\square$

In what follows, Proposition 4.10 is used to verify the tripod property for essential partitions obtained by rearrangements based on building blocks. With this objective in mind, we say that an atom A, which is a pair-wise essentially disjoint union of building blocks, *consists of building blocks.* An essential partition of A into building blocks is not uniquely determined. However, when the essential partition into building blocks should be clear from the context, we denote this adjacency graph by $\widetilde{\Gamma}(A)$. Note that $\widetilde{\Gamma}(A)$ is always a tree.

4.2. **Flat rearrangements.** We first discuss flat local rearrangements using centered building blocks in n-cubes. By local we mean the arrangements occur in a cube of side length 9. We consider different cases, starting from simple and heading to more complicated constructions. The constructions are scale-invariant, but we work with cubes of fixed side length 1, 3, and 9 for simplicity.

Let Q be an n-cube of side length 9 and F a face of Q. We subdivide Q into 3^n congruent n-cubes of side length 3; let $\mathcal{Q}$ be the collection of all these subcubes, i.e. $\mathcal{Q} = Q^*$. Then $\mathcal{Q}$ induces a subdivision of F into 3^{n-1} congruent $(n-1)$-cubes of side length 3. The collection of these $(n-1)$-cubes is denoted by $\mathcal{F}$, i.e. $\mathcal{F} = F^*$, and $\mathcal{Q}(Q; F)$ is the subset of cubes in $\mathcal{Q}$ with a face in $\mathcal{F}$.

Definition 4.13. *A quadruple $(Q, F, \mathcal{Q}_0', q_0)$ forms* initial data *if*

(a) *q_0 is an n-cube of side length 3 so that $q_0 \cap Q$ is a face of q_0 and $q_0 \cap F$ is an $(n-2)$-cube, and*
(b) *$\mathcal{Q}_0' \subset \mathcal{Q}(Q; F)$ is a collection with*
  (i) *$\Gamma(\mathcal{Q}_0')$ connected and*
  (ii) *$q_0 \cap |\mathcal{Q}_0'| = q_0 \cap Q$.*

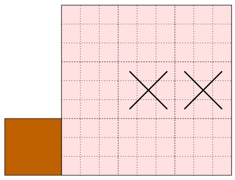

FIGURE 17. An example of an initial data $(Q, F, \mathcal{Q}_0', q_0)$. The face F and cube q_0 viewed from above, cubes in $\mathcal{Q}(Q; F) \setminus \mathcal{Q}_0'$ marked with 'x'; $n = 3$.

Definition 4.14. *Let $(Q, F, \mathcal{Q}_0', q_0)$ be initial data. A maximal tree $\Gamma \subset \Gamma(\mathcal{Q}_0' \cup \{q_0\})$ is a* spanning tree associated to this initial data *if Γ has valence less than $2(n-1)$.*

The valence bound $2(n-1)$ in Definition 4.14 stems from the valence bound for building blocks, see Definition 4.7.

The following simple lemma shows the existence of spanning trees in the configurations we consider here. Let q_F be the unique cube of side length 3 in $\mathcal{Q}(Q; F)$ having valence $2(n-1)$ in $\Gamma(\mathcal{Q}(Q; F))$; note that the barycenter of $q_F \cap F$ is the barycenter of F.

Lemma 4.15. *Suppose $(Q, F, \mathcal{Q}_0', q_0)$ forms initial data and $\Gamma(\mathcal{Q}_0' \setminus \{q_F\})$ is connected. Then there exists a spanning tree $\Gamma \subset \Gamma(\mathcal{Q}_0')$.*

Proof. Let Γ' be a maximal tree in $\Gamma(\mathcal{Q}_0' \setminus \{q_F\})$. Since $\Gamma(\mathcal{Q}_0' \setminus \{q_F\}) \subset \Gamma(\mathcal{Q}(Q; F))$ and q_F is the unique vertex in $\Gamma(\mathcal{Q}(Q; F))$ having valence $2(n-1)$, Γ' is a spanning tree of $\Gamma(\mathcal{Q}_0' \setminus \{q_F\})$. If $q_F \notin \mathcal{Q}_0'$, we may take $\Gamma = \Gamma'$.

If $q_F \in \mathcal{Q}_0'$, our hypothesis produces $q' \in \Gamma(\mathcal{Q}_0')$ adjacent to q_F, so we extend Γ' to a tree Γ containing q_F by adding the edge $\{q', q_F\}$. Since the valence of q' in Γ' is less than $2(n-1) - 1$, the claim follows. $\square$

Spanning trees repartition Q using atoms; recall the adjacency graph $\widetilde{\Gamma}(A)$ of building blocks in an atom A introduced at the end of Section 4.1.

Lemma 4.16. *Given initial data $(Q, F, \mathcal{Q}_0', q_0)$ and a spanning tree Γ, there exists a 1-fine atom A_Γ in Q with the following properties:*

(1) *$A_\Gamma \cap q'$ is an F-based building block for every $q' \in \mathcal{Q}_0'$,*
(2) *the adjacency graph $\widetilde{\Gamma}(A_\Gamma)$ of building blocks is $\Gamma \setminus \{q_0\}$,*
(3) *$A_\Gamma \cup q_0$ is an n-cell, and*
(4) *$A_\Gamma \cap \partial Q \subset F \cup q_0$.*

We call A_Γ the *(unique) atom associated with spanning tree Γ (and initial data $(Q, F, \mathcal{Q}_0', q_0)$).*

Remark 4.17. *Note that atom A_Γ in Lemma 4.16 is on the boundary of Q as defined in Section 3.1. Thus $Q - A_\Gamma$ is a dented cube and, in particular, an n-cell.*

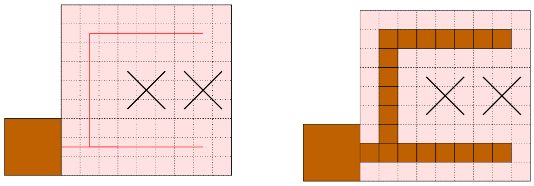

FIGURE 18. A spanning tree (left) and the corresponding atom (right) associated to the initial data in Figure 17.

Proof of Lemma 4.16. To obtain the building blocks, we make the following observation.

Suppose $q' \in \Gamma$ is a vertex other than q_0. Let $\Gamma_{q'}$ be the star of q' in Γ, that is, the subgraph of Γ containing only edges connecting to q' and all vertices on these edges. We denote $E_{q'} = |\Gamma_{q'}|$. Then $E_{q'}$ is a building block.

To each $q' \in \mathcal{Q}'_0$ corresponds a unique F-based centered building block $B_{q'} \subset q'$ which is a translation of $(1/3)E_{q'}$. Let A_Γ be the 1-fine F-based atom for which $\{B_{q'} \colon q' \in \mathcal{Q}'_0\}$ is an essential partition; $A_\Gamma = \bigcup_{q' \in \mathcal{Q}'_0} B_{q'}$. Then the adjacency graph $\tilde{\Gamma}(A) = \Gamma(\{B_{q'} \colon q' \in \mathcal{Q}'_0\})$ is isomorphic to Γ.

Conditions (1), (2), and (4) are clearly satisfied by the construction. Since Γ is a tree, A_Γ is an atom. Since q_0 is a leaf in Γ and $A_\Gamma \cap q_0$ is an $(n-1)$-cube, $A_\Gamma \cup q_0$ is an n-cell and (3) holds. □

Atoms associated to initial data and spanning trees immediately yield a local tripod property.

Lemma 4.18. *Let Q and Q' be n-cubes of side length 9 sharing the face F. Suppose $(Q, F, \mathcal{Q}(Q;F), q_0)$ forms initial data with spanning tree Γ. Let A_Γ be the atom associated to Γ and $(Q, F, \mathcal{Q}(Q;F), q_0)$. Then the essential partition $\mathbf{U} = (Q - A_\Gamma, A_\Gamma, Q')$ of $Q \cup Q'$ has the tripod property.*

Proof. Let q be a cube in $\mathcal{Q}(Q;F)$ and let q_- be the unique cube in Q' sharing a face with q. Denote by B_q the building block $q \cap A_\Gamma$. By Proposition 4.10, $(q - B_q, B_q, q_-)$ satisfies the tripod property. Let Δ_q be an essential partition of $(\partial_\cup \mathbf{U}) \cap q$ as in Definition 4.4. Since $\{q_-\} \cup \mathcal{Q}(Q;F)$ is an essential partition of a cubical set having $\partial_\cup \mathbf{U}$ (essentially) in its interior, $\Delta = \bigcup_{q \in \mathcal{Q}(Q;F)} \Delta_q$ is a required essential partition of $\partial_\cup \mathbf{U}$. □

More generally, we may consider initial data $(Q, F, \mathcal{Q}'_0, q_0)$, where $q_0 \in \mathcal{Q}(Q;F)$; then $q_0 \subset Q$ with $q_0 \cap F$ a face of q_0. Initial data of this type is called *internal initial data*. This notion of initial data is especially useful for extending a 3-fine building block inside a cube of side length 9. We formulate now this rearrangement procedure.

Corollary 4.19. *Let Q be a cube of side length 9, F a face of Q. For $1 \le r \le p$, let q_r be pair-wise essentially disjoint cubes in $\mathcal{Q}(Q;F)$ and suppose each $(Q, F, \mathcal{Q}'_r, q_r)$ forms internal initial data for which $\mathcal{Q}'_r \subset \mathcal{Q}(Q;F)$ and $\mathcal{Q}'_t \cap \mathcal{Q}'_s = \varnothing$ for $t \ne s$. Suppose $\Gamma_1, \ldots, \Gamma_p$ are spanning trees for these initial data, respectively. Then there exist pair-wise disjoint 1-fine atoms A_r associated to initial data $(Q, F, \mathcal{Q}'_r, q_r)$ for $r = 1, \ldots, p$.*

It is easy to obtain a local tripod property for these repartitions. We leave the details, similar to those of the proof of Lemma 4.18, to the interested reader.

Corollary 4.20. *Let Q and Q' be n-cubes of side length 9 sharing the face F, and suppose that, for $1 \le r \le p$, $(Q, F, \mathcal{Q}'_r, q_r)$ forms internal initial data as in*

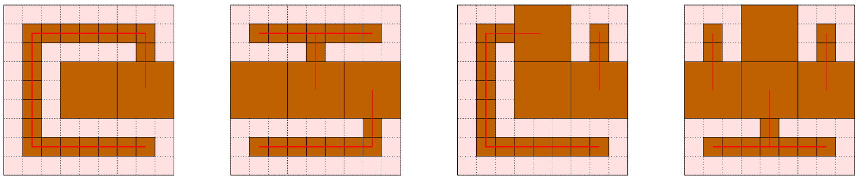

FIGURE 19. Some examples of atoms A_r for $r = 1, \dots, p$ and $p = 1, 2, 3$.

Corollary 4.19, together with the additional property that

$$B := |\mathcal{Q}(Q; F)| - \bigcup_{r=1}^{p} |\mathcal{Q}'_r|$$

is a building block of side length 3. Suppose Γ_p is a spanning tree for $(Q, F, \mathcal{Q}'_r, q_r)$ for every $1 \le r \le p$. Then the essential partition

$$\mathbf{U} = (Q - (B \cup A), B \cup A, Q')$$

of $Q \cup Q'$ has the tripod property, where A is the disjoint union of atoms A_r associated initial data $(Q, F, \mathcal{Q}'_r, q_r)$ and spanning trees Γ_r for $1 \le r \le p$.

Convention. *Henceforth we do not differentiate between initial data and internal initial data, and refer to both as initial data.*

4.3. **Non-flat rearrangements.** We consider now local rearrangements in the non-flat case. For our purposes it suffices to consider rearrangements which occur in a single cube.

Let Q be an n-cube of side length 9 and $\mathcal{F}$ a subset of faces of Q. Let $\mathcal{F}$ be partitioned into sets $\mathcal{F}^1$ and $\mathcal{F}^2$ so that $|\mathcal{F}^r|$ is an $(n-1)$-cell for $r = 1, 2$.

Let $\mathcal{Q}(Q; \mathcal{F}) \subset Q^*$ the cubes having a face in $|\mathcal{F}|$; we denote by $\mathcal{Q}(Q; \mathcal{F}^r) \subset Q^*$ those with a face in $|\mathcal{F}^r|$. Note that $\{\mathcal{Q}(Q; \mathcal{F}^1), \mathcal{Q}(Q; \mathcal{F}^2)\}$ is not (necessarily) a partition of $\mathcal{Q}(Q; \mathcal{F})$. The following definition generalizes Definition 4.13.

Definition 4.21. *A triple*

$$(Q, (\mathcal{F}^1, \mathcal{Q}''_1, q_1), (\mathcal{F}^2, \mathcal{Q}''_2, q_2))$$

forms non-flat initial data *if the following conditions are satisfied:*

(a) *for every $r = 1, 2$, $q_r \subset \mathbb{R}^n - Q$ is an n-cube of side length 3 with $Q \cap q_r$ a face of q_r and $q_r \cap |\mathcal{F}^r|$ an $(n-2)$-cube.*
(b) *$\{\mathcal{Q}''_1, \mathcal{Q}''_2\}$ is a partition of $\mathcal{Q}(Q; \mathcal{F})$ and for $r = 1, 2$ satisfies*
 (0) *$\mathcal{Q}''_r \subset \mathcal{Q}(Q; \mathcal{F}^r)$,*
 (1) *$\Gamma(\mathcal{Q}''_r)$ is connected,*
 (2) *$q_r \cap |\mathcal{Q}''_r|$ is a face of q_r, and*
 (3) *$q_r \cap |\mathcal{F}^r| \cap |\mathcal{Q}''_r|$ is an $(n-2)$-cube.*

Remark 4.22. *Let $(Q, (\mathcal{F}^1, \mathcal{Q}''_1, q_1), (\mathcal{F}^2, \mathcal{Q}''_2, q_2))$ be as in Definition 4.21, and let q_1^+ and q_2^+ be the n-cubes in Q^* sharing a face with q_1 and q_2, respectively. Since q_1 meets Q only in one face, condition (2) in (b) shows that $q_1^+ \in \mathcal{Q}''_1$. Furthermore, by (a), q_1 has at least one face in $|\mathcal{F}^1|$. Similarly, $q_2^+ \in \mathcal{Q}''_2$ and q_2^+ has at least one face in $|\mathcal{F}^2|$.*

Let $r \in \{1, 2\}$, $\widehat{\Gamma} \subset \Gamma(\mathcal{F}^r)$ a maximal tree and $\mathcal{Q}' \subset \mathcal{Q}(Q; \mathcal{F}) \cup \{q_r\}$. A subgraph $\Gamma \subset \Gamma(\mathcal{Q}')$ is *dominated by* $\widehat{\Gamma}$ if, for each vertex $q \in \Gamma$, either there exists a vertex $F_q \in \Gamma(\mathcal{F}^r)$ satisfying $\Gamma_q \setminus \{q_r\} \subset \mathcal{Q}(Q; F_q)$ or there exists an edge $\{F_q, F'_q\} \in \widehat{\Gamma}$ satisfying $\Gamma_q \setminus \{q_r\} \subset \mathcal{Q}(Q; F_q) \cup \mathcal{Q}(Q; F'_q)$; here Γ_q is the star of q in Γ.

Definition 4.23. *Let* $(Q, (\mathcal{F}^1, \mathcal{Q}''_1, q_1), (\mathcal{F}^2, \mathcal{Q}''_2, q_2))$ *form non-planar initial data. A maximal forest* $\Sigma = \Gamma_1 \cup \Gamma_2 \subset \Gamma(\mathcal{Q}'' \cup \{q_1, q_2\})$ *is a* spanning forest associated to this data *if*

(i) Σ *has valence less than* $2(n-1)$*,*
(ii) *for* $r = 1, 2$*,* Γ_r *is a maximal tree in* $\Gamma(\mathcal{Q}''_r \cup \{q_r\})$ *dominated by a maximal tree of* $\Gamma(\mathcal{F}^r)$*.*

The proof of the following existence result for spanning forests is analogous to Lemma 4.15. We omit the details. Let $\mathcal{Q}'_c(Q; \mathcal{F})$ be the collection of all cubes in $\mathcal{Q}(Q; \mathcal{F})$ having valence $2(n-1)$.

Lemma 4.24. *Suppose* $(Q, (\mathcal{F}^1, \mathcal{Q}''_1, q_1), (\mathcal{F}^2, \mathcal{Q}''_2, q_2))$ *forms non-planar initial data for which* $\Gamma(\mathcal{Q}''_r \setminus \mathcal{Q}_c(Q; \mathcal{F}))$ *is connected for* $r = 1, 2$*. Then there exists a spanning forest* Σ *associated to* $(Q, (\mathcal{F}^1, \mathcal{Q}''_1, q_1), (\mathcal{F}^2, \mathcal{Q}''_2, q_2))$*.*

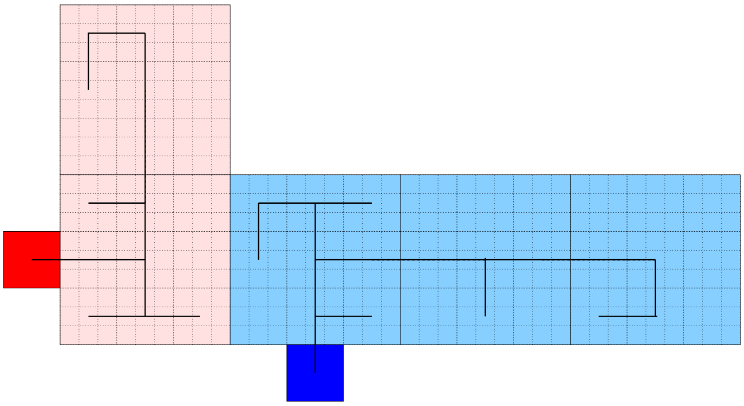

FIGURE 20. A non-planar initial data and a spanning forest on five faces of a cube.

Lemma 4.25. *Let* $(Q, (\mathcal{F}^1, \mathcal{Q}''_1, q_1), (\mathcal{F}^2, \mathcal{Q}''_2, q_2))$ *form non-planar initial data, and let* $\Sigma = \Gamma_1 \cup \Gamma_2 \subset \Gamma(\mathcal{Q}' \cup \{q_1, q_2\})$ *be a spanning forest.*

Then there exist a 1*-fine cubical set* A_Σ *in* Q *composed of pair-wise disjoint* 1*-fine atoms* A_1 *and* A_2 *and for* $r = 1, 2$ *satisfying the following properties:*

(1) *each* A_r *is composed of building blocks,*
(2) *for every* $q'' \in \mathcal{Q}''_r$*,* $A_r \cap q''$ *is an atom having an essential partition into at most two building blocks,*
(3) *every building block in* A_r *is* F*-based with* $F \in \mathcal{F}^r$*,*
(4) $A_r \cup q_r$ *is an* n*-cell,*
(5) $A_r \cap \partial Q \subset |\mathcal{F}^r| \cup q_r$*, and*
(6) *the adjacency graph of cells* $\{A_r \cap Q'' : Q'' \in \mathcal{Q}''_r\}$ *is isomorphic to* Γ_r*.*

The set A_Σ in Lemma 4.25 is said to be *associated to this initial data and the spanning forest* Σ. Property (3) asserts that A_Σ is *on the boundary of* Q.

Remark 4.26. *Similarly as in Remark 4.17, the components* A_1 *and* A_2 *of* A_Σ *in Lemma 4.25 are atoms on the boundary of* Q*. In particularly,* $Q - A_\Sigma$ *is a dented cube.*

Proof of Lemma 4.25. Consider first the tree Γ_1. Let $q' \in \Gamma_1$ be an F-based cube, where $F \in \mathcal{F}^1$, and let $\Gamma_{q'}$ be the star of q' in Γ_1.

If $|\Gamma_{q'}|$ is F-based, we fix a building block $B_{q'}$ as in Lemma 4.16. Suppose, however, that $|\Gamma_{q'}|$ is not F-based. Then, by (ii) in Definition 4.23, there exists a

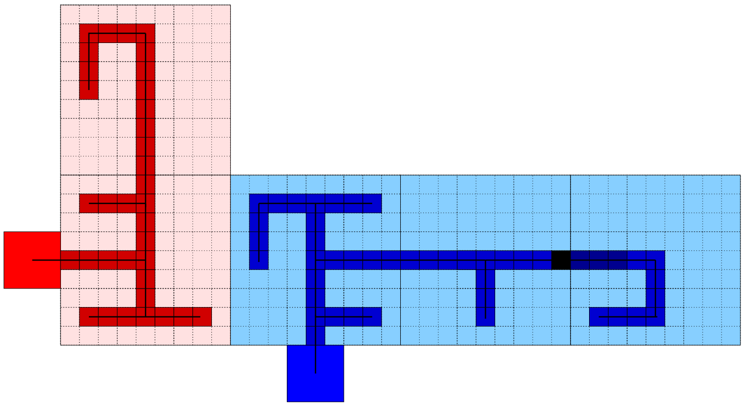

FIGURE 21. Atoms A_1 and A_2 associated to the initial data in Figure 20.

face $F' \in \mathcal{F}^1$ so that each cube in $\Gamma_{q'}$ is either F-based or F'-based. Thus there exist an F-based building block B_F and an F'-based building block $B_{F'}$ in q' with the following properties:

- $B_F \cap B_{F'}$ is an $(n-1)$-cube and
- $B_F \cup B_{F'}$ meets the neighbors of q' in Γ_1 in $(n-1)$-cubes.

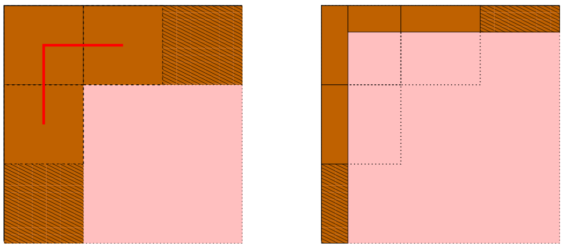

FIGURE 22. The star of q' (left) and building blocks in the star of q' (right).

In this case, we take $B_{q'} = B_F \cup B_{F'}$, and define $A_1 = \bigcup_{q'\in\Gamma} B_{q'}$. The atom A_2 is defined similarly. It is easy to check that atoms A_1 and A_2 satisfy properties (1)-(5). $\square$

These non-planar rearrangements satisfy the tripod property.

Lemma 4.27. *Let $\mathbf{U} = (U_1, U_2, U_3)$ be an essential partition and $Q \subset U_1$ an n-cube of side length 9 sharing a face with both U_2 and U_3. Let*

$$(Q, (\mathcal{F}^1, \mathcal{Q}_1'', q_1), (\mathcal{F}^2, \mathcal{Q}_2'', q_2))$$

form non-planar initial data for which

(i) *for every r there exists $i_r \in \{2,3\}$ so that $q_r \subset U_{i_r}$, and*
(ii) *$|\mathcal{F}^r| \subset Q \cap U_{j_r}$, where $\{j_r, i_r\} = \{2,3\}$, and*
(iii) *$|\mathcal{F}^1| \cup |\mathcal{F}^2| = Q \cap \partial_\cup \mathbf{U}$.*

Let Σ be a spanning forest for this initial data and let $A_\Sigma = A_1 \cup A_2$ be the union atoms associated to this initial data and spanning forest.

Then essential partition

$$\mathbf{V} = \left(U_1 - A_\Sigma, U_2 \cup \bigcup_{i_r=2} A_r, U_3 \cup \bigcup_{i_r=3} A_r \right)$$

has the tripod property.

Proof. It suffices to verify that $\partial_\cup \mathbf{V}$ satisfies the tripod property in every cube in $\mathcal{Q}(Q;\mathcal{F})$.

Let $q \in \mathcal{Q}(Q;\mathcal{F})$. We consider two cases. Suppose first that $b = q \cap A_\Sigma$ is a building block, with A_Σ from Lemma 4.25. Let q' be the unique n-cube in $U_2 \cup U_3$ sharing a side with q. By Proposition 4.10, the essential partition $(q-b, b, q')$ of $q \cup q'$ satisfies the tripod property.

Suppose next that $A = q \cap B_\Sigma$ has an essential partition into two building blocks, say b_1 and b_2. By (ii), there are exactly two n-cubes q_1 and q_2 in $U_2 \cup U_3$ sharing a side with q. Let $f_1 = q \cap q_1$ and $f_2 = q \cap q_2$. By relabeling, we may assume that b_r is f_r-based for $r = 1, 2$. Since the building blocs b_1 and b_2 are centered and do not contain common n-cubes, we may assume, by relabeling again if necessary, that $b_2 \cap f_1 = \varnothing$. Since $b_1 \cup b_2$ is connected, it follows that $c_{bf} = b_1 \cap f_2$ must be an $(n-1)$-cube. We also note that the set $c_{bb} = (\partial b_1) \cap b_2$ is a unit $(n-1)$-cube and $(\partial b_1) \cap b_2 = b_1 \cap (\partial b_2)$. Define $E_1 = (\partial_\cup (q, q - b_1, q_1) - c_{bb}) \cup c_{bf}$ and $E_2 = \partial_\cup (q, q-b_2, q_2) - (c_{bb} \cup c_{bf})$.

Thus, by elementary modifications to the proof of Proposition 4.10, there exists, for $r = 1, 2$, an essential partition Δ_r of E_r satisfying the conditions of Definition 4.4, so that $\Delta = \Delta_1 \cup \Delta_2$ is an essential partition of $\partial_\cup (q, q - A, q_1 \cup q_2)$ satisfying the conditions of Definition 4.4. The claim follows. $\square$

4.4. **Neglected faces in $\mathcal{Q}(Q;\mathcal{F})$.** We finish this section by a slight modification of our analysis for non-flat initial data. To motivate it, consider Figure 21. It is easy to find a cube q in $\mathcal{Q}(Q;\mathcal{F})$ which meets more faces of ∂Q than $q \cap A_\Sigma$; in Figure 21 this phenomenon presents itself as 'empty squares of side length 3'. The next definition formalizes this observation.

Definition 4.28. *Let $(Q, (\mathcal{F}^1, \mathcal{Q}_1'', q_1), (\mathcal{F}^2, \mathcal{Q}_2'', q_2))$ form non-flat initial data, Σ be a spanning forest, and let $A_\Sigma = A_1 \cup A_2$ be the cubical set associated to Σ from Lemma 4.25. We say that $q \in \mathcal{Q}(Q;\mathcal{F})$* has an A_Σ-neglected face f *if $f \subset \partial Q$ while $f \cap A_\Sigma$ has no interior.*

Remark 4.29. *A cube $q \in \mathcal{Q}(Q;\mathcal{F})$ has a neglected face if and only if q has more faces contained in ∂Q than $q \cap A_\Sigma$ has building blocks.*

Denote by $\mathcal{N}(Q;A_\Sigma)$ the collection of all A_Σ-neglected faces in cubes in $\mathcal{Q}'$. Furthermore, for $p = 1, 2$, let $\mathcal{N}_p(Q;A_\Sigma)$ be the collection of all faces in $\mathcal{N}(Q;A_\Sigma)$ contained in $|\mathcal{F}^p|$.

Definition 4.30. *Suppose $q \in \mathcal{Q}(Q;\mathcal{F})$ has an A_Σ-neglected face f and let $p \in \{1,2\}$ be such that $f \subset |\mathcal{F}^p|$. We say that f* admits a (flat) extension of A_Σ *if there exists $q' \in \mathcal{Q}(Q;\mathcal{F})$ adjacent to q and a face f' of q' contained in $|\mathcal{F}^p|$ so that $q' \cap A_p$ contains an f'-based atom and $f \cap f'$ is an $(n-2)$-cube. We call f' a* link into f.

Denote by $\mathcal{N}_{\mathrm{ext}}(Q;A_\Sigma)$ the collection of all faces in $\mathcal{N}(Q;A_\Sigma)$ admitting an extension of A_Σ.

To extend atoms over all neglected faces, we first introduce the notion of a pre-basin.

Definition 4.31. *Given $p \in \{1,2\}$, a collection $C \subset \mathcal{N}(Q;A_\Sigma)$ is a* pre-basin on $|\mathcal{F}^p|$ *if*

(PB1) *$|C| \subset F$ for some $F \in \mathcal{F}^p$,*
(PB2) *$\Gamma(C)$ is connected, and*
(PB3) *$C \cap \mathcal{N}_{\mathrm{ext}}(Q;A_\Sigma) \neq \varnothing$,*

Let $C \subset \mathcal{N}(Q; A_\Sigma)$ be a pre-basin. By (PB3), we may fix $f_C \in C \cap \mathcal{N}_{\mathrm{ext}}(Q; A_\Sigma)$. Let f'_C be a link into f_C, and denote by q_C and q'_C the unique cubes in $\mathcal{Q}(Q; \mathcal{F})$ having f_C and f'_C as faces, respectively. Having this link in mind, we denote by σ_C the connected component of $f'_C - A_\Sigma$, chosen so $\sigma_C \cap f_C$ contains an $(n-2)$-cube. Let also $\Omega_C = |C| \cup \sigma_C$. The cell Ω_C is called an *extension of* $|C|$ *to* f'_C.

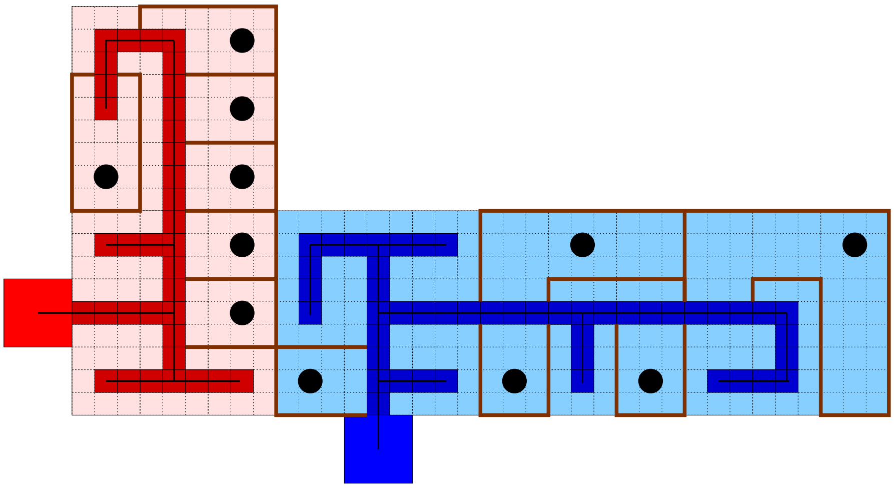

FIGURE 23. Extended pre-basins for a partition of $\mathcal{N}(Q; A_\Sigma)$ into pre-basins given the data in Figure 21. Pre-basins indicated with dots.

This formulation of pre-basins is sufficient for all forthcoming constructions in dimensions $n > 3$. When $n = 3$ we will also need to subdivide pre-basins. We formalize this with the notion of system of basins; however this procedure is (quite) general and need not be restricted only to dimension $n = 3$.

Let $\mathfrak{P}$ be a partition of $\mathcal{N}(Q; A_\Sigma)$ into pre-basins, and suppose we have fixed, for each $C \in \mathfrak{P}$, an extension Ω_C of $|C|$; see Figure 23. We denote $\Omega_{\mathfrak{P}} = \bigcup_{C \in \mathfrak{P}} \Omega_C$.

Definition 4.32. *An essential partition* $\mathcal{B}$ *of* $\Omega_{\mathfrak{P}}$ *is a* system of basins (associated to $\Omega_{\mathfrak{P}}$) *if*

(B1) *each* $B \in \mathcal{B}$ *is a subset of* $F \in \mathcal{F}^1 \cup \mathcal{F}^2$*,*
(B2) $\Gamma(B^\#)$ *is connected for every* $B \in \mathcal{B}$
(B3) $\Gamma(B^\#)$ *admits a spanning tree,*
(B4) $B \cap A_\Sigma$ *contains a unit* $(n-2)$*-cube for every* $B \in \mathcal{B}$*,*
(B5) *for every* $B \in \mathcal{B}$ *there exists* $C \in \mathfrak{P}$ *so that* $B - |\mathcal{N}(Q; A_\Sigma)| \subset \sigma_C$*.*

The elements of $\mathcal{B}$ *are called* basins.

Note that condition (B5) is more flexible than requiring that $B - |C| \subset \sigma_C$.

Remark 4.33. *The existence of a system of basins is straightforward given a partition* $\mathfrak{P}$ *of* $\mathcal{N}(Q; A_\Sigma)$*. Indeed, for every* $C \in \mathfrak{P}$*, fix* $f_C \in C \cap \mathcal{N}_{\mathrm{ext}}(Q; A_\Sigma)$*. Let* f'_C *be a link into* f_C *and denote by* σ_C *the connected component of* $f'_C - A_\Sigma$ *as above. We then subdivide* $\bigcup_{C \in \mathfrak{P}} \sigma_C$ *into pair-wise disjoint* 1*-fine sets* σ'_C *with connected graphs* $\Gamma(\sigma'_C)$ *so that the sets* $B_C = |C| \cup \sigma'_C$ *satisfy conditions (B2) and (B4) for every* $C \in \mathfrak{B}$*. Since* $\Gamma(\sigma'_C)$ *has valence less than* $2(n-1) - 1$ *and* $|C|$ *is* 3*-fine, it is also straightforward to show that* $\Gamma(B^\#_C)$ *admits a spanning tree. Clearly conditions (B1) and (B5) are satisfied. Thus* $\mathcal{B} = \{B_C : C \in \mathfrak{P}\}$ *is a system of basins.*

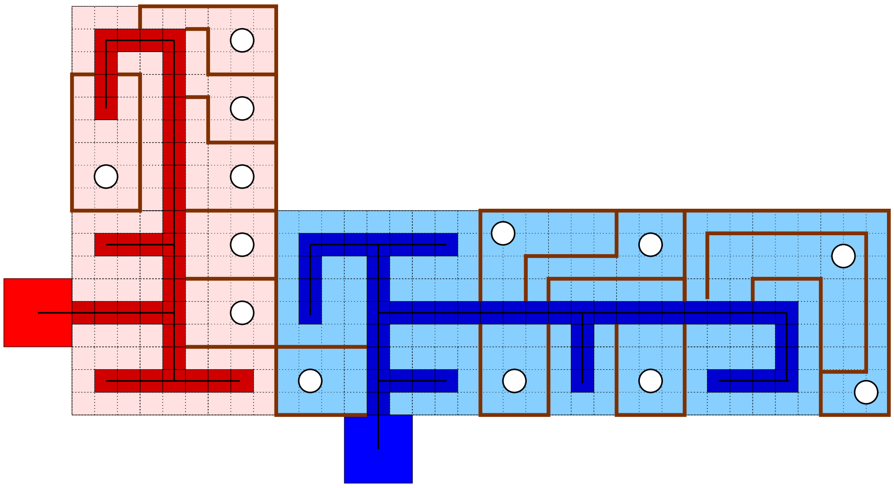

FIGURE 24. A partition of $\mathcal{N}(Q; A_\Sigma)$ into basins associated to the data of Figure 23. Basins indicated with dots.

Finally, we introduce a (flat) rearrangement along a system of basins. Let $\mathcal{B}$ be a system of basins associated to $\Omega_{\mathfrak{P}}$, and let $B \in \mathcal{B}$. By (B5) we may fix $q_B \in A_\Sigma$ so that $B \cap q_B$ is an $(n-2)$-cube. This cube q_B is called the *link of A_Σ into B*.

Let $F_B \in \mathcal{F}^1 \cup \mathcal{F}^2$ be the unique face of Q satisfying (B1). Then the quadruple $(3Q, 3F_B, 3B^\#, 3q_B)$ satisfies the conditions for flat initial data with the only exception that $3Q$ and F_B now have side length 27. We call $(3Q, 3F_B, 3B^\#, 3q_B)$ *scaled flat initial data.*

By (B3), we may fix, for every $B \in \mathcal{B}$, a spanning tree Γ_B of $\Gamma(3B^\# \cup \{3q_B\})$. Similarly, as in the proof of Lemma 4.16, we find an atom A_{Γ_B} associated with the initial data $(3Q, 3F_C, 3B^\#, 3q_B)$ and the spanning tree Γ_B. This observation is formalized as the next lemma, with the details left to the interested reader.

Lemma 4.34. *Let Q be a cube of side length 9 and $A_\Sigma \subset Q$ a union of two atoms as in Lemma 4.25. Suppose $\mathcal{B}$ is a system of basins associated to $\Omega_{\mathfrak{P}}$, where $\mathfrak{P}$ is a partition of $\mathcal{N}(Q; A_\Sigma)$ into pre-basins. For every $B \in \mathcal{B}$, let $(3Q, 3F_C, 3B^\#, 3q_B)$ be a scaled flat initial data and Γ_B a spanning tree of $\Gamma(3B^\# \cup \{3q_B\})$.*

Then there exist 1-fine pair-wise disjoint atoms A_{Γ_B}, $B \in \mathcal{B}$, satisfying conditions (1)-(5) in Lemma 4.16 and so that $3A_\Sigma \cup \bigcup_{B\in\mathcal{B}} A_{\Gamma_B}$ is a pair-wise disjoint union of two molecules.

5. Rough Rickman partitions

In this section we produce a rough Rickman partition $(\tilde\Omega_1, \tilde\Omega_2, \tilde\Omega_3)$ of $\mathbb{R}^n$, i.e. prove Proposition 1.4 in the introduction. The proof is based on the existence of uniform essential partitions associated to the exhaustion of $[0,\infty)^{n-1} \times \mathbb{R} = \bigcup_{k\ge 0} 3^k Q_0$, where $Q_0 = [0,1]^{n-1} \times [-1,1]$.

Proposition 5.1. *For $m \ge 0$, there exist essential partitions*

$$\mathbf{\Omega}_m = (\Omega_{m,1}, \Omega_{m,2}, \Omega_{m,3})$$

of n-cells $3^m \left(Q_0 \cup ([1,2] \times [0,1]^{n-1}\right)$ with the following properties:

(1) *the sequence $(\mathbf{\Omega}_m)$ is stable in the sense that:*
   (1a) $\Omega_{m,p} \cap 3^{m-2}Q_0 = \Omega_{m',p} \cap 3^{m-2}Q_0$ *for $m' > m > 2$ and $p = 1, 2, 3$,*
   (1b) $\Omega_{m,3} \subset \operatorname{int} [0,\infty)^{n-1} \times \mathbb{R} = \bigcup_{m\ge 0} \mathbf{\Omega}_m$*;*
(2) *each $\Omega_{m,j}$ is a dented molecule satisfying*

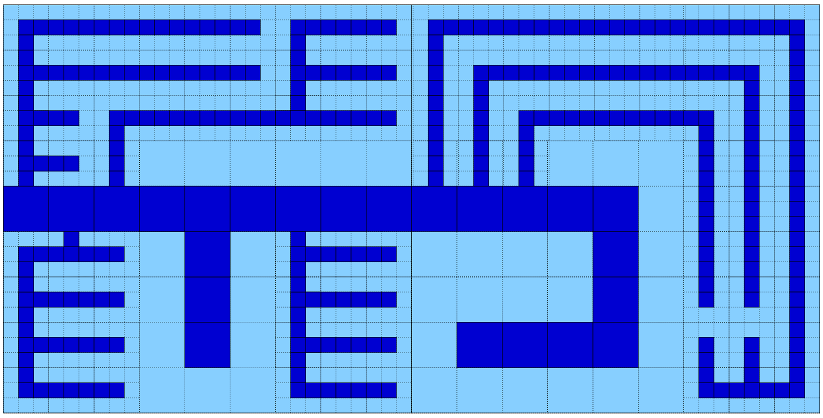

FIGURE 25. A partial close-up of $3A_\Sigma \cup \bigcup_{B\in\mathcal{B}} A_{\Gamma_B}$ associated to A_Σ in Figure 24.

(2a) *there exist* $\nu \ge 1$, $\delta \in (0,1)$, *and* $\ell_0 \ge 1$ *depending only on* n *so that each* $\mathrm{hull}(\Omega_{m,j})$ *is a* (ν,δ)*-molecule with atom length at most* ℓ_0 *and*

(2b) *there exist* $L \ge 1$ *depending only on* n *and an* L*-bilipschitz homeomorphism* $(\Omega_{m,j}, d_{\Omega_{m,j}}) \to (\mathrm{hull}(\Omega_{m,j}), d_{\mathrm{hull}(\Omega_{m,j})})$ *which is the identity on* $\partial\mathrm{hull}(\Omega_{m,j}) \cap \Omega_{m,j}$*;*

(3) *every* $\mathbf{\Omega}_m$ *satisfies the tripod property.*

Furthermore, for $p = 1,2,3$, *the domain* $\Omega_p = \bigcup_{m\ge 0} \Omega_{m,p}$ *in its inner metric* d_{Ω_p} *is bilipschitz equivalent to* $\mathbb{R}^{n-1}\times[0,\infty)$*. Indeed, there exist bilipschitz homeomorphisms* $\phi_1 \colon [0,\infty)^{n-1}\times[0,\infty) \to (\Omega_1, d_{\Omega_1})$ *and* $\phi_2 \colon [0,\infty)^{n-1}\times(-\infty,0] \to (\Omega_2, d_{\Omega_2})$ *which restrict to the identity mappings on* $\partial[0,\infty)^{n-1}\times[0,\infty)$ *and* $\partial[0,\infty)^{n-1}\times(-\infty,0]$*, respectively; the boundary* $\partial[0,\infty)^{n-1}$ *is understood in* $\mathbb{R}^{n-1}$*.*

Conditions (1)-(3) have the following interpretations. Condition (1) refers to the induction process, which consists of two main steps: scaling and rearranging, and allows us to paste the essential partitions $\mathbf{\Omega}_m$ together. The closed sets $\Omega_{m,3}$ are contained in the interior of $[0,\infty)^{n-1}\times\mathbb{R}$. Condition (2) yields that the domains $\Omega_{m,j}$ are uniformly bilipschitz equivalent to cubes $[0,3^m]^n$. Finally, (3) ensures that $\mathrm{dist}_{\mathcal{H}}(\partial_\cup\mathbf{\Omega}_m, \partial_\cap\mathbf{\Omega}_m) \le 6$ in the sup-metric; compare with (4.3).

Proof of Proposition 1.4 given Proposition 5.1. Let $\mathbf{\Omega}' = (\Omega_1', \Omega_2', \Omega_3')$ be the essential partition of $[0,\infty)^{n-1}\times\mathbb{R}$ from Proposition 5.1. By (1a) and (3) in Proposition 5.1, $\mathbf{\Omega}'$ satisfies the tripod property.

We subdivide $\mathbb{R}^n$ into 2^{n-1} congruent subsets $W_1,\ldots,W_{2^{n-1}}$, where $W_1 = [0,\infty)^{n-1}\times\mathbb{R}$. Since $\Omega_3' \subset \mathrm{int}\, W_1$, we obtain, by reflecting Ω_3' with respect to the common sides of $W_1,\ldots,W_{2^{n-1}}$, pair-wise disjoint domains $\Omega_4',\ldots,\Omega_{2^{n-1}+2}'$. We denote by Ω_1 and Ω_2 the unions of the corresponding reflections of Ω_1' and Ω_2'; both Ω_1 and Ω_2 are connected.

Let $\psi_1' \colon [0,\infty)^{n-1}\times[0,\infty) \to (\Omega_1', d_{\Omega_1'})$ and $\psi_2' \colon [0,\infty)^{n-1}\times(-\infty,0] \to (\Omega_2', d_{\Omega_2'})$ be bilipschitz homeomorphisms which reduce to the identity mapping on the boundary as in Proposition 5.1. Reflections across the pair-wise common sides of domains $W_1,\ldots,W_{2^{n-1}}$ extend ψ_1' and ψ_2' to bilipschitz homeomorphisms $\psi_1 \colon \mathbb{R}^{n-1}\times[0,\infty) \to (\Omega_1, d_{\Omega_1})$ and $\psi_2 \colon \mathbb{R}^{n-1}\times(-\infty,0] \to (\Omega_2, d_{\Omega_2})$. Finally, if

$$\Omega_3 = \Omega_3' \cup \cdots \cup \Omega_{2^{n-1}+2}',$$

and

$$\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3),$$

(3) ensures that $\mathbf{\Omega}$ is a rough Rickman partition satisfying the tripod property. □

5.1. **$\mathcal{C}$-, $\mathcal{D}$-, and $\mathcal{E}$-modifications.** In this section we introduce three key modifications related the iterative scaling and rearrangement process which gives Proposition 5.1. These are particular applications of the rearrangements in cubes of side length 9 introduced in Section 4, and they form a sufficiently rich class to achieve Proposition 5.1. The main purpose of the rearrangements used here is to achieve the tripod property in Proposition 5.1.

Let $\mathbf{U} = (U_1, U_2, U_3)$ be an essential partition, E an n-cell in $|\mathbf{U}|$, and denote $\mathbf{U} \cap E = (U_1 \cap E, U_2 \cap E, U_3 \cap E)$. Similarly, $\mathbf{U} - E = (U_1 - E, U_2 - E, U_3 - E)$.

Let Ω be a 3-fine n-cell and suppose that $\mathbf{U} = (U_1, U_2, U_3)$ is an essential partition of Ω into n-cells. A cube $Q \in \Omega^*$ of side length 3 is a $\mathbf{U}$-*cube* if there exists $i \in \{1, 2, 3\}$ for which $Q \subset U_i$. The index i is the *color of* Q *in* $\mathbf{U}$, and the indices $\{1, 2, 3\} \setminus \{i\}$ are *complementary indices (of the color of* Q*)*. Denote by $\mathcal{Q}_\partial(\mathbf{U})$ the cubes Q in Ω^* with $Q \cap \partial_\cup \mathbf{U}$ containing an $(n-1)$-cell.

5.1.1. *$\mathcal{C}$- and $\mathcal{D}$-cubes.* We introduce first $\mathcal{C}$- and $\mathcal{D}$-cubes; the letters $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ informally suggest *cube*, *dent*, and *external*, respectively. After presenting these definitions, we discuss the corresponding rearrangements. Let U be an n-cell and $\mathbf{U} = (U_1, U_2, U_3)$ an essential partition of U.

Suppose Q is a $\mathbf{U}$-cube of color $i \in \{1, 2, 3\}$, and let j and k be complementary indices. For $p = j, k$, let $\mathcal{Q}'_p(Q)$ be the collection of all unit n-cubes in $Q^\#$ meeting U_p in an $(n-1)$-cube, and denote $\mathcal{Q}'(Q) = \mathcal{Q}'_j(Q) \cup \mathcal{Q}'_k(Q)$. Let $\mathcal{Q}'_c(Q)$ be the cubes in $\mathcal{Q}'(Q)$ having valence $2(n-1)$ in the adjacency graph $\Gamma(\mathcal{Q}'(Q))$.

Let $Q \in \mathcal{Q}_\partial(\mathbf{U})$ be a $\mathbf{U}$-cube of color i and side length 3, and suppose for complementary indices j and k there are unit n-cubes $q_j \subset U_j$ and $q_k \subset U_k$ with $q_j \cap q_k = \varnothing$ and both cubes q_j and q_k having a face contained in ∂Q; let q'_j and q'_k be the unique cubes in $\mathcal{Q}'(Q)$ which share a face with q_j and q_k, respectively.

Definition 5.2. *In this situation,* Q *is a* $\mathcal{C}$*-cube in* $\mathbf{U}$ *if for* $\{p, r\} = \{j, k\}$*, the adjacency graph* $\Gamma(\{q_r\} \cup (\mathcal{Q}'_p(Q) \setminus (\mathcal{Q}'_c(Q) \cup \{q'_p\})))$ *is connected.*

The collection of $\mathcal{C}$-cubes in $\mathbf{U}$ is denoted as $\mathcal{C}(\mathbf{U})$. Note that each $Q \in \mathcal{C}(\mathbf{U})$ satisfies $Q \cap \partial_\cup \mathbf{U} \subset \partial Q$, since $\mathcal{C}$-cubes are $\mathbf{U}$-cubes.

Suppose $\mathbf{V} = (V_1, V_2, V_3)$ and $\mathbf{U} = (U_1, U_2, U_3)$ are essential partitions of Ω.

Definition 5.3. *A cube* $Q \in \mathcal{Q}_\partial(\mathbf{U})$ *of side length* 3 *is a* $\mathcal{D}$-cube in $\mathbf{U}$ relative to $\mathbf{V}$ *if* Q *is a* $\mathbf{V}$*-cube of color* i *but not a* $\mathbf{U}$*-cube, and there exist complementary indices* $j \neq k$ *with*

(1) $A := Q \cap U_j$ *either a* $(Q \cap \partial V_i)$*-based building block in* Q *or a single atom made of two building blocks based on different faces of* Q*, and*
(2) $(Q - A, A, \Omega)$ *has the tripod property, where* Ω *is the smallest* n*-cell consisting of* n*-cubes of side length* 3 *for which* $A \cap \partial Q \subset \Omega$*.*

The collection of all $\mathcal{D}$-cubes in $\mathbf{U}$ with respect to $\mathbf{V}$ is called $\mathcal{D}(\mathbf{U}; \mathbf{V})$. Note that A in (1) is always a 1-fine atom. If A in (1) is a building block, we say that Q is a *$\mathcal{D}$-cube of type* 1. Otherwise, Q is a *$\mathcal{D}$-cube of type* 2.

5.1.2. *$\mathcal{D}$-modifications.* We consider first $\mathcal{D}$-cubes of type 1. The next rearrangement is called a *$\mathcal{D}$-modification.*

Lemma 5.4. *Let* $\mathbf{V} = (V_1, V_2, V_3)$ *and* $\mathbf{W} = (W_1, W_2, W_3)$ *be essential partitions of the same* n*-cell by* n*-cells, and let* $Q \in \mathcal{D}(\mathbf{V}; \mathbf{W})$ *be a* $\mathcal{D}$*-cube of type* 1*; let* i *be the color of* Q *in* $\mathbf{W}$ *and let* j *be such that* $A := Q \cap V_j$ *is an* F*-based building block, where* F *is face of* Q*. Then there exists a pair-wise disjoint union of building blocks* $B_\Sigma \subset 3Q$ *composed of* 1*-fine* F*-based atoms on the boundary of* $3Q$*, so that*

the n-cells $U_i = 3V_i - B_\Sigma$, $U_j = 3V_j \cup B_\Sigma$, and $U_k = 3V_k$, where k is the other complementary index, form an essential partition

$$\mathbf{U} = (U_1, U_2, U_3)$$

of $|3\mathbf{V}|$ satisfying

(1) $B_\Sigma \cap \partial(3Q) \subset \partial_\cup 3\mathbf{V}$,
(2) $\partial_\cup \mathbf{U} \cap 3Q \subset |\mathcal{C}(\mathbf{U})| \cup |\mathcal{D}(\mathbf{U}; 3\mathbf{V})|$, *and*
(3) B_Σ *is an atom for $n > 3$ and consists of at most 3 components for $n = 3$.*

Furthermore, if $Q \cap \partial W_i \subset F$, then $\mathbf{U}$ satisfies the tripod property in $3Q$.

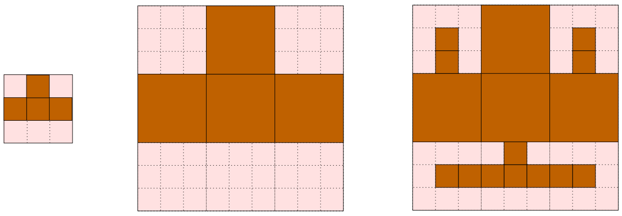

FIGURE 26. An example of an essential partition $\mathbf{V}$ in Q, and essential partitions $3\mathbf{V}$ and $\mathbf{U}$ in $D = 3Q$ for one example of a building block $Q \cap W_j$ shown in Figure 19.

Convention. *Before proceeding to the proof of Lemma 5.4, we emphasize that the figures in this section (e.g. in Figure 26 above) use only the two complementary colors j and k. The third color, the color i of the cube itself, never appears.*

Proof of Lemma 5.4. It suffices to find flat initial data for Corollary 4.19, the claim then follows from Lemma 4.18. Note that Corollary 4.19 is necessary only for $n = 3$ and for $n > 3$ we may use Lemma 4.16. Let $D = 3Q$.

Define $F' = 3F \cap 3V_i$, that is, $F' = 3F - 3A$. For $n > 3$, F' is connected. For $n = 3$, F' consists of at most three 2-cells.

Let D^* be the 3-regular subdivision of D and let $(D^*)' \subset D^*$ be the subset of cubes having a face contained in F'. For $n > 3$ we fix a unit cube Q_1 in A. Then $Q_1 \cap F'$ is an $(n-2)$-cell. For $n = 3$, we fix unit cubes $Q_1, \ldots, Q_p$ in A, where p is the number of components of F'.

When $n > 3$ we choose a maximal tree $\Sigma \in \Gamma((D^*)' \cup \{Q_1\})$, and for $n = 3$, we fix a maximal forest $\Sigma = \Gamma_1 \cup \cdots \cup \Gamma_p$ in $\Gamma((D^*)' \cup \{Q_1, \ldots, Q_p\})$. The vertex sets of trees Γ_i give now a required partition for $(D^*)'$. Corollary 4.19 yields a 1-fine set B_Σ whose components are $3F$-based atoms. The tripod property in D for $\mathbf{U}$ follows from Lemma 4.18 and $B_\Sigma \subset F \cup \operatorname{int} D$ by condition (4) in Lemma 4.16.

Assertions (1) and (2) follow by Corollary 4.19, the fact that cubes in $(D^*)'$ are $\mathcal{D}$-cubes in $\mathcal{D}(\mathbf{U}; 3\mathbf{V})$ and the observation that $3A = |\mathcal{C}(\mathbf{U})| \cap D$. $\square$

For $\mathcal{D}$-cubes of type 2, the corresponding arrangement is also called a *$\mathcal{D}$-modification.*

Lemma 5.5. *Let $\mathbf{V} = (V_1, V_2, V_3)$ and $\mathbf{W} = (W_1, W_2, W_3)$ be essential partitions of the same n-cell with n-cells, and $Q \in \mathcal{D}(\mathbf{V}; \mathbf{W})$ be a $\mathcal{D}$-cube of type 2; let i be the color of Q in $\mathbf{W}$ and take j so that $A := Q \cap V_j = B \cup B'$ is an atom, where B and B' are essentially disjoint building blocks. Then there exists a pair-wise disjoint union $A_\Sigma \subset 3Q$ of 1-fine atoms on the boundary of $3Q$ consisting of building blocks with*

$$\mathbf{U} = (U_1, U_2, U_3)$$

an essential partition of $|3\mathbf{V}|$ *by* n*-cells satisfying the tripod property in* $3Q$*. Here* $U_i = 3V_i - A_\Sigma$*,* $U_j = 3V_j \cup A_\Sigma$*,* $U_k = 3V_k$ *and* k *is the remaining complementary index. Moreover,*

(1) $A_\Sigma \cap \partial(3Q) \subset \partial_\cup 3\mathbf{V}$*, and*
(2) $\partial_\cup \mathbf{U} \cap 3Q \subset |\mathcal{C}(\mathbf{U})| \cup |\mathcal{D}(\mathbf{U}; 3\mathbf{V})|$*, and*
(3) A_Σ *is an atom for* $n > 3$ *and consists of at most 4 components for* $n = 3$*.*

Moreover, if $Q \cap \partial W_i \subset f \cup f'$*, where* f *and* f' *are the adjacent faces of* Q*, satisfying* $A \cap \partial W_i \subset f \cup f'$*, then* $\mathbf{U}$ *has the tripod property in* $3Q$*.*

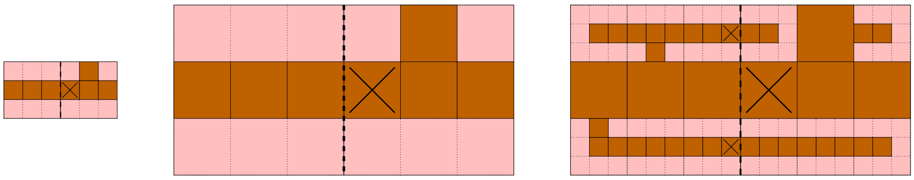

FIGURE 27. Analogue of Figure 26 for $\mathcal{D}$-cube of type 2.

Proof of Lemma 5.5. We may assume $i = 1$, $j = 2$, and $k = 3$ and that B and B' are f- and f'-based, respectively. Let $D = 3Q$.

This case uses Lemma 4.25 in place of Corollary 4.19 and Lemma 4.27 in place of Lemma 4.18.

Let $\mathcal{Q}'$ be the collection of cubes in D^* meeting $f \cup f'$ and not contained in $3A^{\#}$.

Recall that $\Gamma(\mathcal{Q}')$ is the adjacency graph of cubes in $\mathcal{Q}'$. For $n > 3$, $\Gamma(\mathcal{Q}')$ is connected, and we may fix a cube $q \in 3A^{\#}$ of side length 3 and a maximal tree $\Sigma \subset \Gamma(\mathcal{Q}' \cup \{q\})$. It is a simple observation that we may now apply Lemma 4.25 to Σ in place of a forest and obtain a 1-fine atom A_Σ satisfying (1)-(5).

For $n = 3$, we observe that $\Gamma(\mathcal{Q}')$ has at most 4 components $\Gamma_1, \ldots, \Gamma_p$, $p \le 4$; Figure 27 illustrates $p = 3$ when $n = 3$. It is easy to observe that we may fix pairwise essentially disjoint cubes $q_1, \ldots, q_p$ in $3A^*$ so that $\Gamma(\Gamma_i \cup \{q_i\})$ is connected, so we may create $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_p$ a maximal forest with $\Sigma_i \subset \Gamma(\Gamma_i \cup \{q_i\})$. A slight modification of Lemma 4.25 yields $A_\Sigma = A_{\Sigma_1} \cup \cdots \cup A_{\Sigma_p}$, where A_{Σ_i} is a 1-fine atom satisfying (1)-(5).

In both cases, we may check that

$$\mathbf{U} = (3V_1 - A_\Sigma, 3V_2 \cup A_\Sigma, 3V_3)$$

satisfies the required conditions. □

The essential properties of $\mathcal{D}$-modifications are summarized in two corollaries.

Corollary 5.6. *Let* $\mathbf{V}$*,* Q*,* A*,* $\mathbf{U}$*, and* $\{i, j, k\} = \{1, 2, 3\}$ *be as in Lemma 5.4 or as in Lemma 5.5. Then* $\mathbf{U} \cap 3Q$ *satisfies the tripod property as well as*

(a) $\partial_\cup \mathbf{U} \cap 3Q \subset |\mathcal{C}(\mathbf{U})| \cup |\mathcal{D}(\mathbf{U}; 3\mathbf{V})|$ *and*
(b) $\mathcal{C}(\mathbf{U}) = 3(Q \cap W_j)^{\#}$*.*

Furthermore, to each $f \in (((\partial_\cup \mathbf{V}) \cap Q) - A)^{\#}$ *corresponds a* $3f$*-based building block in* U_j*.*

By (1) in Lemmas 5.4 and 5.5, the $\mathcal{D}$-modifications are independent between cubes in $3\mathcal{D}(\mathbf{V}; \mathbf{W})$ in the sense that all $\mathcal{D}$-modifications in adjacent $\mathcal{D}$-cubes agree on common faces of the cubes. This is summarized in the following corollary; the notation $\widetilde{\Gamma}(\cdot)$ has been introduced at the end of Section 4.1.

Corollary 5.7. *Let $\mathbf{V} = (V_1, V_2, V_3)$ and $\mathbf{W} = (W_1, W_2, W_3)$ be essential partitions of the same n-cell by n-cells so that V_p is a dented molecule for $p = 1, 2, 3$. Then there exists an essential partition $\mathbf{U} = (U_1, U_2, U_3)$ of $|3\mathbf{V}|$ into n-cells so that $\mathbf{U}$ satisfies the tripod property in each cube in $3\mathcal{D}(\mathbf{V}; \mathbf{W})$ and*

(a) $\mathbf{U} - |3\mathcal{D}(\mathbf{V}; \mathbf{W})| = 3\mathbf{V} - |3\mathcal{D}(\mathbf{V}; \mathbf{W})|$,
(b) *if $Q \in 3\mathcal{D}(\mathbf{V}; \mathbf{W})$, then $\mathbf{U} \cap Q$ is obtained from $3\mathbf{V}$ by a $\mathcal{D}$-modification,*
(c) *each leaf $A \in \widetilde{\Gamma}(U_i)$ is a 1-fine atom adjacent to a 3-fine atom $A' = 3a'$, where a' is a leaf in $\widetilde{\Gamma}(V_i)$, and*
(d) *for each leaf $a \in \widetilde{\Gamma}(V_i)$ there exists at most $3\ell_{\mathrm{bb}}(\widetilde{\Gamma}(V_i))$ leaves in $\widetilde{\Gamma}(U_i)$ adjacent to $3a \in \widetilde{\Gamma}(U_i)$, where $\ell_{\mathrm{bb}}(\widetilde{\Gamma}(V_i)) = \max_A \ell_{\mathrm{bb}}(\widetilde{\Gamma}(A))$ and the maximum is taken over the leaves A in $\widetilde{\Gamma}(V_i)$.*

5.1.3. *$\mathcal{C}$- and $\mathcal{E}$-modifications.* The next rearrangement is a *$\mathcal{C}$-modification.*

Lemma 5.8. *Let V be an n-cell and $\mathbf{V} = (V_1, V_2, V_3)$ an essential partition of V. Suppose $Q \in \mathcal{C}(\mathbf{V})$ has color i in $\mathbf{V}$. Let j and k be complementary to i in $\{1, 2, 3\}$. Then there exist atoms A_j and A_k in $3Q$ which are composed of building blocks along $\partial(3Q)$ so that $U_i = 3V_i - (A_j \cup A_k)$, $U_j = 3V_j \cup A_j$, and $U_k = 3V_k \cup A_k$ are n-cells and*

$$\mathbf{U} = (U_1, U_2, U_3) \tag{5.1}$$

is an essential partition of $3V$ into n-cells having the tripod property in C. Moreover,

(1) $(A_j \cup A_k) \cap \partial(3Q) \subset \partial_\cup 3\mathbf{V}$ *and*
(2) $(\partial_\cup \mathbf{U}) \cap 3Q \subset |\mathcal{D}(\mathbf{U}; 3\mathbf{V})|$.

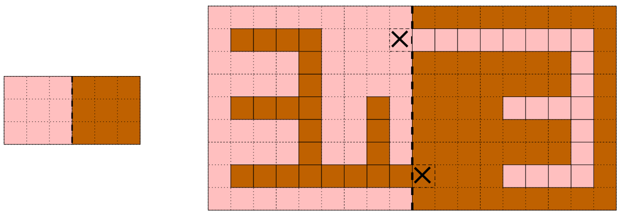

FIGURE 28. Cube Q and essential partition $\mathbf{U}$ in $3Q$.

Proof of Lemma 5.8. The proof is a straightforward application of Lemma 4.18 to appropriate non-flat initial data. Let $C = 3Q$.

For notational convenience, take $i = 3$. Let $q_1 \subset V_1$ and $q_2 \subset V_2$ be unit cubes as in Definition 5.2. For $p = 1, 2$, let $\mathcal{F}^p$ be the collection of faces of Q which meet V_p in an $(n-1)$-cell. Then

$$(Q, (\mathcal{F}^1, \mathcal{Q}(Q; \mathcal{F}^2), q_1), (\mathcal{F}^2, \mathcal{Q}(Q; \mathcal{F}^1), q_2))$$

are non-flat initial data.

Let Σ be a spanning forest as in Lemma 4.24 and A_1 and A_2 be atoms associated to Σ as in Lemma 4.16. By Lemma 4.27, the essential partition $(V_1 \cup A_1, V_2 \cup A_2, V_3 - (A_1 \cup A_2))$ satisfies the tripod property in C.

Property (1) follows immediately from (5) in Lemma 4.25, and (2) from the observation that every cube in $\mathcal{Q}'(Q)$ is a $\mathcal{D}$-cube. $\square$

The $\mathcal{C}$-modification in Lemma 5.8 is a 'primary' $\mathcal{C}$-modification. To illustrate the need for an 'extended' $\mathcal{C}$-modification, consider the following example.

Example 5.9. *Let* $\mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3) = ([0,3]^3, [0,3]^2 \times [-3,0], [3,6] \times [0,3]^2)$. *The cube* $Q = [0,3]^3$ *is a* $\mathcal{C}$*-cube of color* 1 *in* $\mathbf{V}$.

Using Lemma 5.8 we perform a $\mathcal{C}$*-modification in* $C = 3Q$*, that is, obtain the essential partition* $\mathbf{U}$ *relative to* C *as in Lemma 5.8; see Figure 28.*

Consider now the essential partition $3\mathbf{U}$. *An example of an essential partition* $\mathbf{U}'$ *of* $|3\mathbf{U}|$ *is in Figure 29. Notice in Figure 28 that* $\mathbf{U}$ *already has* 3 *neglected faces (Definition 4.28), each of which meets the vertical dashed fold. Thus* $\mathbf{U}'$ *cannot satisfy the tripod property; a glance at Figure 29 also shows that there are now* 3 3*-cubes of side length* 9 *in* $\mathbf{U}'$ *for which neither* $\mathcal{C}$*- nor* $\mathcal{D}$*-modification is applicable.*

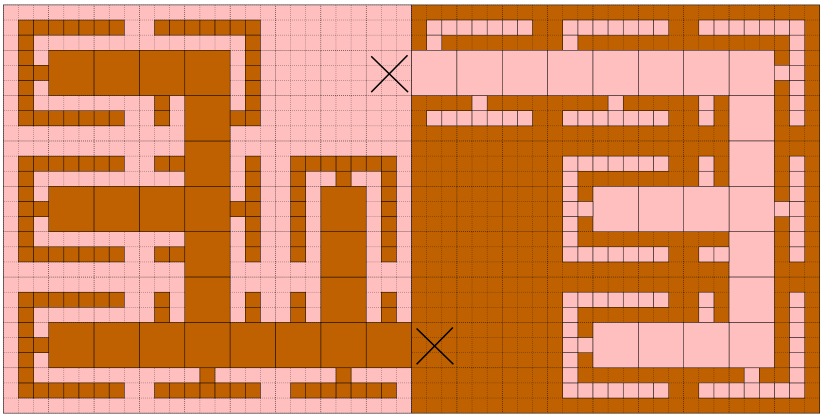

FIGURE 29. Essential partition $\mathbf{U}'$ in $9Q$.

In this situation, we achieve the tripod property by what we call an 'extended $\mathcal{C}$-modification', called an $\mathcal{E}$*-modification*. The proof imitates that of Lemma 5.4, and takes into account both neglected faces and $\mathcal{D}$-cubes. The details are left to the interested reader.

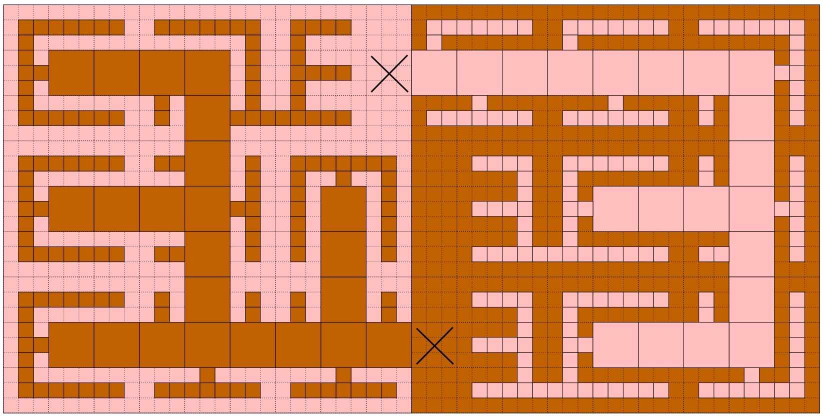

FIGURE 30. An $\mathcal{E}$-modification $\mathbf{U}'$ of the essential partition $\mathbf{U}$ shown in Figure 28.

Lemma 5.10. *Let* V *be an* n*-cell,* $\mathbf{V} = (V_1, V_2, V_3)$ *be an essential partition of* V*, and* $Q \in \mathcal{C}(\mathbf{V})$ *a cube of color* i *in* $\mathbf{V}$. *Let* A_j *and* A_k*, where* $\{i,j,k\} = \{1,2,3\}$*, be atoms in* $3Q$ *and let* $\mathbf{U} = (U_1, U_2, U_3)$ *be the essential partition* $U_i = V_i - (A_j \cup A_k)$*,* $U_j = V_j \cup A_j$*, and* $U_k = V_k \cup A_k$ *from Lemma 5.8.*

Then there exist molecules M_j and M_k in $3Q$ so that $U_i' = U_i - (M_j \cup M_k)$, $U_j' = U_j \cup M_j$, and $U_k' = U_k \cup M_k$ are n-cells and

$$\mathbf{U}' = (U_1', U_2', U_3') \tag{5.2}$$

is an essential partition of $3V$ into n-cells having the tripod property in $3Q$. Moreover,

1. *$(M_j \cup M_k) \cap \partial(3Q) \subset \partial_\cup \mathbf{U}$,*
2. *$(\partial_\cup \mathbf{U}') \cap 3Q \subset |\mathcal{C}(\mathbf{U}')| \cup |\mathcal{D}(\mathbf{U}'; \mathbf{U})|$,*
3. *$A_j \subset M_j$ and $A_k \subset M_k$,*
4. *for $p = j, k$, $M_p - A_p$ consists of pair-wise disjoint 1-fine atoms made of building blocks,*
5. *for $n > 3$ and $p = 1, 2, 3$, each building block in $\tilde{\Gamma}(A_p)$ meets at most one atom in $M_p - A_p$.*

5.1.4. *Modifications in dimension* 3. In dimension $n = 3$, it is easy to exhibit an explicit catalog of $\mathcal{C}$-modifications associated to building blocks. Similarly, the $\mathcal{E}$-modifications can be explicitly illustrated.

Visible faces. Suppose Q is a cube of side length 3 in $\mathbb{R}^3$ and F a face of Q, and B an F-based building block in Q. Having Figure 16 at our disposal, we observe that for every $q \in B^\#$, $q \cap F$ is a unit square and $B \cap (Q - B)$ a 2-cell consisting of at most 4 faces of q.

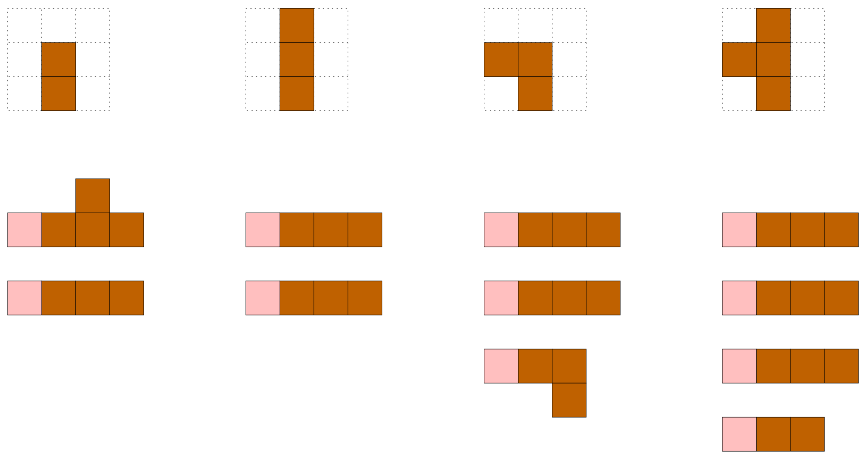

FIGURE 31. Visible faces of building blocks.

Figure 31 displays foldouts of faces of all (unit) cubes q in building blocks B which may occur in Q. Note that the foldout pictures show only faces of cubes q contained in F or $Q - B$. These faces form the *visible* faces; only these are in $\partial_\cup \mathbf{U}$.

$\mathcal{C}$-modifications. Based on the catalog in Figure 31, we observe that in dimension $n = 3$ it suffices to fix 4 $\mathcal{C}$-modifications which can be applied in all cubes in all building blocks of side length 9. The case of 5 visible faces is illustrated in Figure 32. A comprehensive list of examples of $\mathcal{C}$-modifications to cubes with 3 or 4 visible faces is given in Figure 33.

Summary: Let $3B$ be a building block of side length 9 and suppose that in each $Q' \in 3(B^*)$ we have performed one of the $\mathcal{C}$-modifications illustrated in Figures 32 and 33 and let $A_{Q',i} \subset Q'$, $i = 1, 2$, be the corresponding atoms; $\rho(A_{Q',i}) = 1$. Then

- each atom $A_{Q',i}$ consists of at most 20 building blocks and 56 cubes;

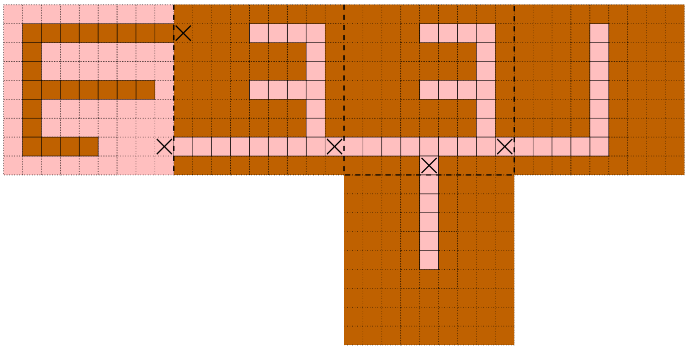

FIGURE 32. Cube q in B with five visible faces.

- in each cube $Q' \in 3(B^*)$, $A_{Q',1} \cup A_{Q',2}$ consists of at most 28 building blocks and 79 cubes;
- $\bigcup_{Q'} (A_{Q',1} \cup A_{Q',2})$ consists of at most 100 building blocks and 285 cubes.

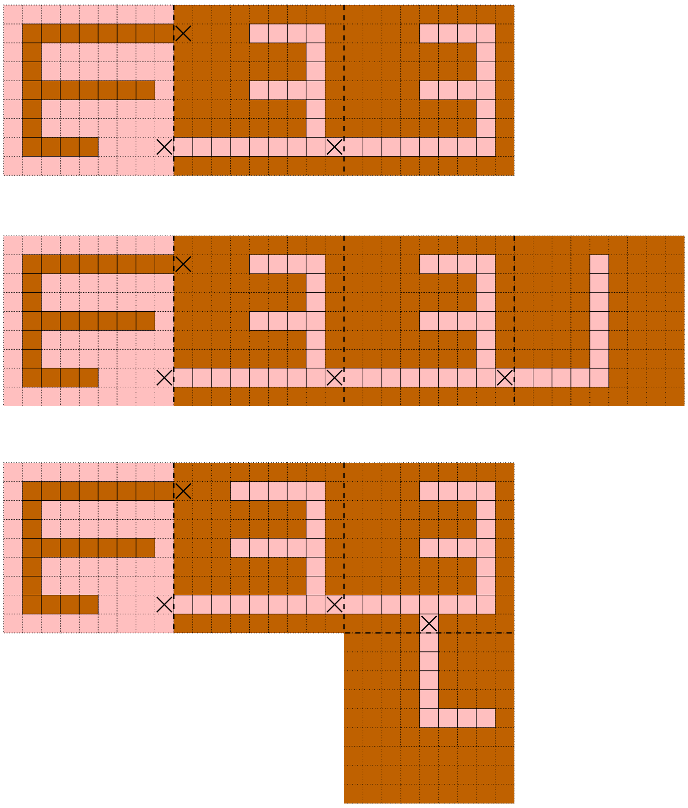

FIGURE 33.

$\mathcal{E}$-modifications. To discuss $\mathcal{E}$-modifications, let Q and B be as above and let Q' be the unique cube sharing the face F with Q. Let $\mathbf{V} = (Q - B, Q', B)$ and let $\mathbf{U} = (U_1, U_2, U_3)$ be the essential partition of $|3\mathbf{V}|$ obtained after $\mathcal{C}$-modifications, based on Figures 32 and 33. Note that components of $3A_1 = U_1 - 3(Q - B)$ are atoms having 8 building blocks.

Let $Q' \in 3(B^*)$. Figure 34 presents an example of a system of basins in Q' when Q' has 5 visible faces. For cubes with fewer visible faces, similar systems

of basins can be found; these systems have fewer basins. Figure 35 illustrates the $\mathcal{E}$-modification in the largest basin in Figure 34. The systems of basins for cubes in $3(B^*)$ with fewer visible faces can be chosen to have basins not larger than this basin in terms of the number of unit cubes in added atoms. We encourage the interested reader to verify these statements by illustrations.

Summary: Let $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ be an essential partition of $3Q$ obtained from $\mathbf{U}$ by an $\mathcal{E}$-modification. For $Q' \in 3(B^*)$, let $M_{Q',j} = \Omega_j \cap Q'$, $j = 1, 2$. Then $M_{Q',j}$ is a molecule having $3A_{Q',j}$ as a root. Let $L_{Q',j} = M_{Q',j} - 3A_{Q',j}$ be the union of leaves of $M_{Q',j}$. Then

- each component of $L_{Q',j}$ consists of at most 16 building blocks and 47 cubes,
- for each cube $Q' \in 3(B^*)$, $L_{Q',1} \cup L_{Q',2}$ has at most 31 components and consists of at most 243 building blocks,
- the union $\bigcup_{Q'} (L_{Q',1} \cup L_{Q',2})$ consists of at most 829 building blocks.

Furthermore, $\Gamma(M_{Q',j})$ has valence at most 45.

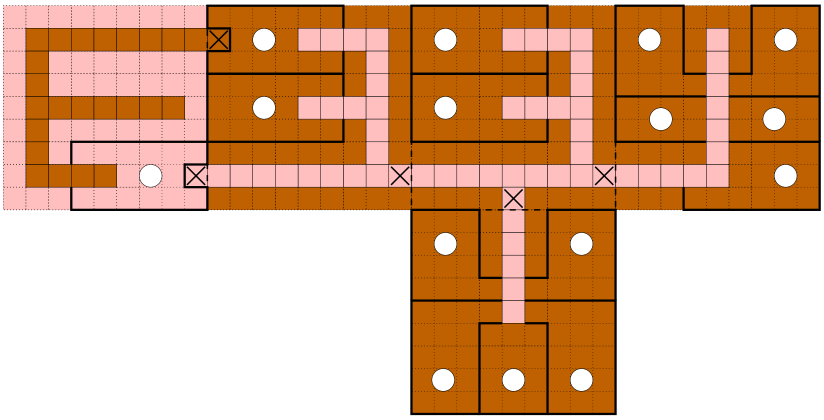

FIGURE 34. An example of a system of basins. Basins indicated with dots.

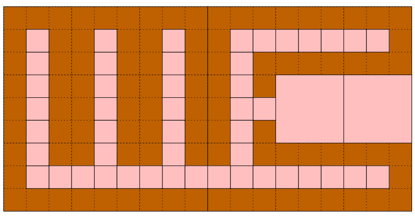

FIGURE 35.

5.2. **Induction with respect to $\mathcal{C}$- and $\mathcal{E}$-modifications.** The next proposition is the inductive step in obtaining the sequence $(\mathbf{\Omega}_m)$ for Proposition 5.1. It is based on scalings and rearrangements provided by $\mathcal{C}$- and $\mathcal{E}$-modifications (Lemmas 5.8 and 5.10, respectively). This induction step does not use $\mathcal{D}$-modifications explicitly; these can be viewed as subrearrangements in $\mathcal{E}$-modifications.

In this induction step, we assume that we are given three essential partitions $\mathbf{V}_0$, $\mathbf{V}_1$, $\mathbf{V}_2$ satisfying certain compatibility conditions. These (initial) partitions will be made concrete in the proof of Proposition 5.1; see forthcoming Section 5.3.

We remind the reader that $\mathcal{C}$- and $\mathcal{D}$-cubes have side length 3 and dimension n, and that $\mathcal{C}$- and $\mathcal{E}$-modifications are performed in appropriate cubes of side length 9 and 27, respectively. The $\mathcal{C}$- and $\mathcal{D}$-cubes were defined in Section 5.1.1.

Proposition 5.11. *Let $n \geq 3$, $\mathbf{V}_0 = ([0,3]^n, [0,3]^{n-1} \times [-3,0], [3,6] \times [0,3]^{n-1})$ and let $\mathbf{V}_1$ and $\mathbf{V}_2$ be essential partitions satisfying the tripod property and the following conditions:*

(1) $|\mathbf{V}_2| = 3|\mathbf{V}_1| = 9|\mathbf{V}_0|$;
(2) $\mathbf{V}_1$ *is obtained by a $\mathcal{C}$-modification in* $[0,9]^n$; *and*
(3) $\mathbf{V}_2$ *is obtained by an $\mathcal{E}$-modification in* $[0,27]^n$.

Then for $m \geq 3$ there exists an essential partition $\mathbf{V}_m = (V_{m,1}, V_{m,2}, V_{m,3})$ which satisfies the tripod property as well as

(a) $|\mathbf{V}_m| = 3|\mathbf{V}_{m-1}|$,
(b) $\partial_\cup \mathbf{V}_m \subset |\mathcal{C}(\mathbf{V}_m)| \cup |\mathcal{D}(\mathbf{V}_m; 3\mathbf{V}_{m-1})| \subset 9|\mathcal{C}(\mathbf{V}_{m-2})|$,
(c) *if $Q \in \mathcal{C}(\mathbf{V}_{m-1})$, $\mathbf{V}_m \cap 3Q$ is obtained from $3(\mathbf{V}_{m-1} \cap Q)$ by a $\mathcal{C}$-modification,*
(d) *if $q \in \mathcal{C}(\mathbf{V}_{m-2})$, $\mathbf{V}_m \cap 9q$ is obtained from $3(\mathbf{V}_{m-1} \cap 3q)$ by an $\mathcal{E}$-modification.*

In addition, there exist $\nu \geq 1$ and $\lambda > 1$, depending only on n, so that for all $m \geq 0$

(e) *each $\mathrm{hull}(V_{m,p})$ is a (ν, λ)-molecule and atom length of $\Gamma(\mathrm{hull}(V_{m,p}))$ is bounded by a constant depending only on n, and*
(f) *there exists $L = L(n) \geq 1$ and an L-bilipschitz map $\psi_{m,p} \colon (\mathbf{V}_{m,p}, d_{V_{m,p}}) \to (\mathrm{hull}(V_{m,p}), d_{\mathrm{hull}(V_{m,p})})$ which is the identity on $V_{m,p} \cap \partial \mathrm{hull}(V_{m,p})$.*

Proposition 5.11 is obtained in three steps. First we construct the sequence $\mathbf{V}_3, \mathbf{V}_4, \ldots$ by induction. Properties (a)–(d) and the tripod property are easy to check directly from the construction and we verify them in Section 5.2.2. Property (e) is more subtle and considered separately in Section 5.2.3. Finally, we prove Property (f), the most demanding part, in Section 5.2.4.

5.2.1. *Inductive construction.* Suppose that essential partitions up to index $m \geq 2$ have been constructed. We then produce an essential partition $\mathbf{V}_{m+1}$.

Let $q \in \mathcal{C}(\mathbf{V}_m)$ and denote $Q = 3q$. Suppose first that $n > 3$. Then, by Lemma 5.8, there exists a $\mathcal{C}$-modification of $3\mathbf{V}_m$ in Q and an essential partition $\mathbf{V}^Q_{m+1}$ of $|3\mathbf{V}_m|$ so that $\mathbf{V}^Q_{m+1} \cap Q$ satisfies the tripod property along with the stability property $\mathbf{V}^Q_{m+1} - Q = 3\mathbf{V}_m - Q$.

For $n = 3$, we fix a suitable $\mathcal{C}$-modification among the examples in Figures 33 and 32 and let $\mathbf{V}^Q_{m+1}$ be the corresponding essential partition in Q, again with $\mathbf{V}^Q_{m+1} - Q = 3\mathbf{V}_m - Q$.

Thus, for each $n \geq 3$, we have that, for every Q and Q' in $3\mathcal{C}(\mathbf{V}_m)$ with non-empty intersection, the essential partitions $\mathbf{V}_{m+1,Q}$ and $\mathbf{V}_{m+1,Q'}$ agree on $Q \cap Q'$. This leads to the essential partition $\mathbf{U} = (U_1, U_2, U_3)$ of $|3\mathbf{V}_m|$ with

$$U_i = (3V_{m,i} - 3|\mathcal{C}(\mathbf{V}_m)|) \cup \bigcup_{Q \in 3\mathcal{C}(\mathbf{V}_m)} V^Q_{m+1,i} \cap Q$$

for $i = 1, 2, 3$.

Suppose now that $q \in \mathcal{C}(\mathbf{V}_{m-1})$ and denote $Q = 9q$. Then, by Lemma 5.10, there exists an $\mathcal{E}$-modification of $3\mathbf{V}_m$ in Q. In dimension $n = 3$, we also use $\mathcal{E}$-modifications satisfying the combinatorial bounds discussed in Section 5.1.4. Based on this observation, we construct $\mathbf{V}_{m+1}$ from $\mathbf{U}$ by rearranging $\mathbf{U}$ in each cube of $9\mathcal{C}(\mathbf{V}_{m-1})$. We leave the details to the interested reader.

5.2.2. *Conditions (a)–(d) and the tripod property.* Clearly, $\mathbf{V}_{m+1}$ is an essential partition of $|3\mathbf{V}_m|$ into n-cells. Furthermore, $\mathbf{V}_{m+1}$ satisfies conditions (a), (c), and (d) by construction.

To check (b) for $\mathbf{V}_{m+1}$, let $\mathcal{Q} = \mathcal{C}(\mathbf{V}_m) \cup \mathcal{D}(\mathbf{V}_m; 3\mathbf{V}_{m-1})$ and note that $\partial_\cup \mathbf{V}_{m+1}$ is covered by cubes in $3\mathcal{Q}$. Thus, for every $Q \in 3\mathcal{Q}$,

$$\begin{aligned} \partial_\cup \mathbf{V}_{m+1} \cap Q &\subset |\mathcal{C}(\mathbf{V}_{m+1} \cap Q)| \cup |\mathcal{D}(\mathbf{V}_{m+1} \cap Q; 3\mathbf{V}_m \cap Q)| \\ &= (|\mathcal{C}(\mathbf{V}_{m+1})| \cup |\mathcal{D}(\mathbf{V}_{m+1}; 3\mathbf{V}_m)|) \cap Q \end{aligned}$$

by conclusion (2) in Lemmas 5.4, 5.5, 5.8, or 5.10 applied to the relevant modification. This verifies (b).

To see that $\mathbf{V}_{m+1}$ has the tripod property we observe first that $\partial_\cup \mathbf{V}_{m+1} - |3\mathcal{Q}| = \varnothing$. Given $Q \in 3\mathcal{Q}$, we may fix an essential partition Δ_Q of $\partial_\cup \mathbf{V}^Q_{m+1} \cap Q$ into $(n-1)$-cells as in Definitions 4.4 and 4.5. Then the collection $\Delta = \bigcup_{Q \in 3\mathcal{Q}} \Delta_Q$ is an essential partition of $\partial_\cup \mathbf{V}_{m+1}$ as required in Definition 4.4, and so $\mathbf{V}_{m+1}$ satisfies the tripod property.

5.2.3. *Condition (e).* We consider first some general properties of dented molecules $V_{m,p}$ and their hulls $\mathrm{hull}(V_{m,p})$, and then prove condition (e) in Proposition 5.11.

Recall from Section 3.2 that a vertex $D \in \Gamma(V_{m,p})$ is internal if there exists a vertex $D' \in \Gamma(V_{m,p})$ so that $D \subset \mathrm{hull}(D')$, and that a vertex is external if it is not internal.

Lemma 5.12. *Suppose A is a leaf in $\Gamma(V_{m,p})$ of side length 3^k and D the vertex adjacent to A in $\Gamma(V_{m,p})$ satisfying $\rho(D) > \rho(A)$. Then $3^{-k}A$ is a leaf of $V_{m-k,p}$.*

If $\rho(D) > 3\rho(A)$, the atom $3^{-k}A$ is obtained by a $\mathcal{C}$-modification and A is an internal vertex of $\Gamma(V_{m,p})$,

Otherwise, $\rho(D) = 3\rho(A)$ and $3^{-k}A$ is obtained by an $\mathcal{E}$-modification. Furthermore, in this case, A is an external vertex of $\Gamma(V_{m,p})$ if and only if D is an external vertex of $\Gamma(V_{m,p})$.

Remark 5.13. *Note that, whereas the number of atoms A attached to D in Lemma 5.12 satisfying $\rho(D) = 3\rho(A)$ is uniformly bounded, this does not hold for atoms A with $\rho(D) > 3\rho(A)$.*

Proof. Since A is a leaf, it is an atom and $k \in \{0, 1\}$. Moreover, $\rho(D) \geq 3\rho(A)$ by construction.

First observe that if $3^{-k}A$ is obtained by a $\mathcal{C}$-modification, there exists a dent M of $3^{-k}D$ with $A \subset M$. Since $M \subset \mathrm{hull}(3^{-k}D)$, it follows that $A \subset \mathrm{hull}(D)$. Thus in this case A is internal and $\rho(A) \leq 3^{-4}\rho(D)$.

Since the ratio of side lengths in an $\mathcal{E}$-modification is 3, the atom $3^{-k}A$ is obtained by an $\mathcal{E}$-modification if and only if $\rho(D) = 3\rho(A)$.

Suppose now that $\rho(D) = 3\rho(A)$. Then $3^{-k}A$ is obtained by an $\mathcal{E}$-modification. We next show that A is an external vertex if and only if D is an external vertex.

Suppose first that A is an internal vertex. Then there exists $D' \in \Gamma(V_{m,p})$ containing A in its hull. Since A is obtained by an $\mathcal{E}$-modification, we have $D \neq D'$. Let M' be the dent of D' containing A. Since A belongs to the component of $V_{m,p} - D'$ contained in D', connectedness of $V_{m,p}$ forces $D \subset M'$. Thus D is internal.

Suppose now that D is an internal vertex. Then there exists $D_0 \in \Gamma(V_{m,p})$ with $D \subset \mathrm{hull}(D_0)$. We may assume that D_0 is minimal in the sense that, for every $D' \in \Gamma(V_{m,p})$ between D and D_0 in $\Gamma(V_{m,p})$, $D \not\subset \mathrm{hull}(D')$. Since $D \subset \mathrm{hull}(D_0)$, there exists a dent M_0 of D_0 for which $D \subset M_0$.

Let D_1 be the unique vertex in $\Gamma(V_{m,p})$ adjacent to D_0 between D and D_0. Since $D \subset M_0$, connectedness of $V_{m,p}$ yields that $D_1 \subset M_1$. Thus the atom $\rho(\mathrm{hull}(D_1))^{-1}\mathrm{hull}(D_1)$ is obtained by $\mathcal{C}$-modification in a cube Q of side length 9.

Let $D_1, \ldots, D_\ell = D$ be the geodesic path in $\Gamma(V_{m,p})$ from D_1 to D. By minimality of D_0, we have, for every $1 \leq k < \ell$, that $\rho(\mathrm{hull}(D_k)^{-1}\mathrm{hull}(D_k)$ is obtained by

an $\mathcal{E}$-modification and in particular, $\rho(D_k) = 3\rho(D_{k+1})$. It is now easy to observe that $A \cup D \subset 3^{\ell-1}Q \subset M_0$. Thus A is an internal vertex. □

Lemma 5.14. *There exist $\nu \geq 1$ and $\lambda < 1$ depending only on n so that the adjacency tree $\Gamma(\mathrm{hull}(V_{m,p}))$ is a (ν, λ)-molecule for every $m \geq 2$ and each p.*

Proof. By Lemma 3.19, $\Gamma(\mathrm{hull}(V_{m,p}))$ is isomorphic to the tree $\Gamma_E(V_{m,p})$ of external vertices of $\Gamma(V_{m,p})$. By Lemma 5.12, external vertices arise from $\mathcal{E}$-modifications. Thus it suffices to estimate the number of atoms created by an $\mathcal{E}$-modification for $m > 2$.

Suppose first that $n > 3$, let $1 < k < m$, and let A be an atom in $\Gamma(V_{k,p})$ created by an $\mathcal{E}$-modification. Then A has side length 1 and it is contained in a union of at most two cubes of side length 9. Since there exist 3^n essentially disjoint cubes of side length 3 in a cube of side length 9, the atom A consists of strictly less than $2 \cdot 3^n$ building blocks; see Remark 5.15 below. Thus an $\mathcal{E}$-modification to $3A$ attaches strictly less than $2 \cdot 3^n$ atoms. We conclude that $\Gamma(\mathrm{hull}(V_{m,p}))$ is at most $(2 \cdot 3^n)$-valent.

To show that $\mathrm{hull}(V_{m.p})$ is λ-collapsible for some $\lambda < 1$, let $M \in \Gamma(\mathrm{hull}(V_{m,p}))$ be a molecule of side length 3^k. Then M is attached to at most $2 \cdot 3^n$ molecules of side length 3^{k-1} and to one molecule M' of side length 3^{k+1}. Let F' be the face of a cube in M' where M and M' meet.

Let $\varepsilon > 0$ to be fixed in a moment, and take ℓ with $(1+\varepsilon)3^{k-1}\ell \leq 3^{k+1} < (1+\varepsilon)3^{k-1}(\ell+1)$. Then there exist at least ℓ^{n-1} pair-wise disjoint $(n-1)$-cubes of side length $(1+\varepsilon) \cdot 3^{k-1}$ on F. Since

$$\ell^{n-1} > \left(\frac{9}{1+\varepsilon} - 1\right)^{n-1}, \tag{5.3}$$

we may fix $\varepsilon > 0$ small enough, depending on n, so that

$$\ell^{n-1} > 2 \cdot 3^n \tag{5.4}$$

for $n \geq 4$. We conclude that M, and hence $\mathrm{hull}(V_{m,p})$, is λ-collapsible with λ depending only on n.

For $n = 3$, we have, by the statistics in Section 5.1.4, that $\Gamma(\mathrm{hull}(V_{m,p}))$ has valency at most 20 and every atom in $\Gamma(\mathrm{hull}(V_{m,p}))$ consists of at most 56 cubes. Since $9^2 > 8^2 > 56$, we may take $\varepsilon = 8/9$ in the argument above. This concludes the proof. □

Remark 5.15. *Note that, although estimates* (5.3) *and* (5.4) *hold also for $n = 3$, the number of building blocks in atom A is not an upper bound for atoms attached to $3A$. In fact, $\mathcal{D}$-modification may attach up to* 3 *atoms to a single building block when $n = 3$.*

5.2.4. *Condition (f).* It suffices to consider $m \geq 4$. Let $p \in \{1, 2, 3\}$. To simplify notation, let $V = V_{m,p}$.

Lemma 5.16. *There exist $L = L(n) \geq 1$ and an L-bilipschitz map $\varphi\colon (V, d_V) \to (\mathrm{hull}(V), d_{\mathrm{hull}(V)})$ which is the identity on $V \cap \partial\mathrm{hull}(V)$.*

We begin the proof of Lemma 5.16 with two auxiliary lemmas. For the statements, we need some new notation and also use terms from Section 3.2.

Let $D \in \Gamma(V)$ and let $D' \in \Gamma(V)$ be the unique vertex adjacent to D with $\rho(D') > \rho(D)$. Let Q_D and Q'_D be the unique cubes of side length $\rho(D)$ in D and D' having a common face F'_D, and F_D the unique face of Q_D sharing an $(n-2)$-cube with F'_D. We call $J_D = F_D \star \{x_{Q_D}\}$ and $J'_D = F'_D \star \{x_{Q'_D}\}$ the *internal* and *external join of D*, respectively. Note that $J_D \subset D$ and $J'_D \subset D'$.

The first key ingredient in the proof of Lemma 5.16 is the following bilipschitz equivalence property for expanding children; recall Definition 3.20.

Lemma 5.17. *Let P be a partial hull of V and let $D \in \Gamma(P)$ be a dented atom having only expanding children. Then there exist $L = L(n) \geq 1$ and an L-bilipschitz map $\varphi_D\colon (|\Gamma(P)_D|, d_{|\Gamma(P)_D|}) \to (\mathrm{hull}(D), d_{\mathrm{hull}(D)})$ which is the identity on $D \cap \partial\mathrm{hull}(D)$ with $\varphi_D(|\Gamma(P)_d|) \subset J_d$, where $d \in \Gamma(P)$ is a child of D.*

Proof. By Lemma 5.14, for every child $d \in \Gamma(P)$, $|\Gamma(P)_d|$ is a collapsible (ν,λ)-molecule with ν and λ depending only on n. Thus, by Proposition 3.5, there exist $L' = L'(n) \geq 1$ and an L'-bilipschitz mapping $\psi_D\colon (|\Gamma(P)_D|, d_{|\Gamma(P)_D|}) \to (D, d_D)$ which is the identity on $D - \bigcup_d J_d$, where the union is over the children of D.

By Proposition 3.12, there exists $L'' = L''(n) \geq 1$ and an L''-bilipschitz map $\phi_D\colon (D, d_D) \to (\mathrm{hull}(D), d_{\mathrm{hull}}(D))$ which is the identity on $D \cap \partial\mathrm{hull}(D)$. Furthermore, by a simple modification of the proof of Proposition 3.12, we may assume that ϕ_D is an isometry from J'_d to J_d for each child d of D. Thus $\varphi_D = \phi_D \circ \psi_D$ is the desired map. □

The second key ingredient in the proof of Lemma 5.16 is the regrouping of joins associated to expanding children of large relative side length. We begin by counting the number of children, and again need some notation.

Let P be a partial hull of $V = V_{m,p}$. Let $D \in \Gamma(P)$ be a dented atom and $B \in \widetilde{\Gamma}(\mathrm{hull}(D))$ a building block. Write $\mathrm{Dent}(D) = \mathrm{hull}(\mathrm{hull}(D) - D)$, and denote by $\mathcal{A}(P, D; B)$ the vertices of $\Gamma(P)$ adjacent to D, which have side length $3^{-4}\rho(D)$ and are contained in B. Note that there are no vertices adjacent to D and contained in B with side length greater than $3^{-4}\rho(D)$.

Lemma 5.18. *Let $D \in \Gamma(P)$ and $B \in \tilde{\Gamma}(\mathrm{hull}(D))$. Then*

$$\#\mathcal{A}(P, D; B) \leq \begin{cases} 8n^2 3^n, & n > 3 \\ 285, & n = 3. \end{cases} \tag{5.5}$$

Proof. The argument is similar to the counting argument in proof of Lemma 5.14. Let $\rho(D) = 3^k$. Let $M_B = B \cap \mathrm{hull}(\mathrm{hull}(D) - D)$. Then M_B is a pair-wise disjoint union of two molecules with $\rho(M_B) = 3^{k-2}$. Let U_B the union of the atoms of side length 3^{k-2} in $\Gamma(M_B)$.

The dented molecules in $\mathcal{A}(P, D; B)$ are in one-one correspondence with $\Gamma^{\mathrm{int}}(U_B)$. Indeed, other cubes in $\Gamma^{\mathrm{int}}(M_D)$ have side length at most 3^{k-3} and the dented molecules adjacent to D which they contain have side length at most 3^{k-5}.

For $n \geq 4$, we have for each $Q \in \Gamma(B)$ the estimate

$$\#\Gamma^{\mathrm{int}}(U_B \cap Q) \leq 2 \cdot 2n \cdot 3^n = 4n3^n,$$

and so

$$\#\Gamma^{\mathrm{int}}(U_B) \leq 2n \cdot 4n3^n = 8n^2 \cdot 3^n.$$

When $n = 3$, the summary in Section 5.1.4 yields that $\#\Gamma^{\mathrm{int}}(U_B) \leq 285$. □

Let $D \in \Gamma(P)$ be a dented molecule and $B \in \widetilde{\Gamma}(\mathrm{hull}(D))$ a building block. Denote by $Q'_B \in \Gamma(B)$ the center of B and by F'_B the unique face of Q'_B contained in $\partial\mathrm{hull}(D')$. Let $Q_B \subset 3^{k-1}(3^{-k}Q'_B)^{\#}$ be the unique cube of side length $3^{-1}\rho(B)$ having $F_B = Q_B \cap F'_B$ as a face of Q_B with the same barycenter as F'_B. We call $J_B = F_B \star \{x_{Q_B}\}$ the *join associated to B*.

Lemma 5.19. *Let P be a partial hull of V. Suppose $D \in \Gamma(P)$ is a dented atom and $A \in \Gamma(\mathrm{hull}(D))$ is the (unique) atom of side length $\rho(D)$. Let Q be the smallest cube having D on the boundary.*

Then there exist an $L = L(n) \geq 1$ and an L-bilipschitz map

$$\psi_D\colon (D, d_D) \to (\mathrm{hull}(D), d_{\mathrm{hull}(D)})$$

which is the identity on $D - A$ and for every $B \in \widetilde{\Gamma}(A)$ satisfies

(1) *$\varphi_D(B) = B$ and*
(2) *for each $d \in \mathcal{A}(P, D; B)$, $\varphi_D|J_d$ is an isometric embedding from J_d into J_B.*

In addition, if for every $B \in \widetilde{\Gamma}(A)$, f_B is an $(n-1)$-cube of side length $3^{-4}\rho(B)$ in $B \cap \partial Q$ having distance at least 3^{-4} to $\partial B - \partial Q$ and to each J_d, then $\varphi_D|f_B \star \{x_{q_B}\}$ is an isometry into J_B, where x_{q_B} is the barycenter of the unique cube q_B in Q having f_B as a face.

Proof. The argument is similar to the collapsing argument in Lemma 5.14. Let $\rho(D) = 3^k$ and $B \in \widetilde{\Gamma}(A)$.

Since $\rho(F_B) = 3^{k-1}$, we may subdivide F_B into 26^{n-1} cubes of side length $(27/26)3^{k-4}$. According to Lemma 5.18,

$$\frac{\#\mathcal{A}(P, D; B)}{26^{n-1}} < 1.$$

Since $\rho(J_d) = 3^{-4}\rho(D) = 3^{k-4}$, there exists for each $d \in \mathcal{A}(P, D; B)$ an $(n-1)$-cube $F''_d \subset F_B$ of side length 3^{k-4} so that the pair-wise distances of these $(n-1)$-cubes are at least $(1/26)3^{k-4}$. Thus there exist $L = L(n) \geq 1$ and an L-bilipschitz map $\psi_B\colon B \to B$ which is the identity on $B - \partial Q$ and which is an isometric embedding from J_d to $F''_d \star \{x_{q''_d}\}$, where q''_d the unique n-cube in Q having F''_d as a face.

The claim now follows by composing these maps. We leave the modification of the argument in the case of additional $(n-1)$-cubes f_B for the interested reader. □

Proof of Lemma 5.16. Let $P_0 = V$ and $\mathcal{J}_0 = \varnothing$. Suppose that, for $k \geq 0$, we have constructed

- partial hulls $P_0, \ldots, P_k$ of V so that $P_{\ell+1}$ is a partial hull of P_ℓ for $0 \leq \ell < k-1$;
- collections $\mathcal{J}_0, \ldots, \mathcal{J}_k$ of joins associated to building blocks in these partial hulls so that joins $\mathcal{J}_\ell$ are contained in atoms of $\Gamma(P_\ell)$ which are hulls of dented atoms D in $\Gamma(P_{\ell-1})$ for $1 \leq \ell \leq k$, and for such D, the number of joins contained in $|\Gamma(P_\ell)_D - D|$ is at most 3^n when $n > 3$ and at most 16 when $n = 3$;
- for every $1 \leq \ell < k$, an L-bilipschitz map $\psi_\ell\colon (P_\ell, d_{P_\ell}) \to (P_{\ell+1}, d_{P_{\ell+1}})$, which is the identity on those atoms of $\Gamma(P_\ell)$ which are atoms in $\Gamma(P_{\ell-1})$, and L is at most the product of bilipschitz constants in Lemmas 5.17 and 5.19.

If $P_k \neq \mathrm{hull}(V)$, we construct P_{k+1} as follows. Since $\Gamma(V)$ is finite, this process terminates.

Since $P_k \neq \mathrm{hull}(V)$, there exist dented atoms in $\Gamma(P_k)$. Let $D_k \in \Gamma(P_k)$ be the dented atom having smallest side length. Let $d \in \Gamma(P_k)$ be an atom adjacent to D_k in $\mathrm{hull}(D_k)$. Then d is expanding.

Let $\mathcal{J}_k(D_k)$ be the joins in $\mathcal{J}_k$ which are contained in $|\Gamma(P_k)_{D_k} - D_k|$. We treat these joins as (virtual) adjacent atoms. Thus every join $J \in \mathcal{J}_k(D_k)$ increases (virtually) the valence of $\Gamma(P_k)_{D_k}$ by 1 at the dented atom containing it. Thus, for $n \geq 4$, the valence of $\Gamma(P_k)_{D_k}$ increases at every vertex by at most 3^n. For $n = 3$, this increase of valence is at most 16 and the maximal (virtual) valence of $\Gamma(P_k)_{D_k}$ is at most $31 + 16 = 47$ since the atoms in $\Gamma(P_k)_{D_k} - D_k$ are expanding and hence obtained by an $\mathcal{E}$-modification or by an $\mathcal{C}$-modification over one face of a cube; see the summary in Section 5.1.4 for these statistics. We leave to the interested reader

to verify that $\Gamma(P_k)_{D_k}$ is λ-collapsible with λ depending only on n even when joins $\mathscr{J}_k(D_k)$ are understood as (virtual) adjacent atoms; compare to Lemma 5.14.

Let $\varphi_k\colon (|\Gamma(P_k)_{D_k}|, d_{|\Gamma(P_k)_{D_k}|}) \to (D_k, d_{D_k})$ be a bilipschitz map as in Lemma 5.17 with the property that, for each $J \in \mathscr{J}_k(D_k)$, $\varphi_k|J$ is an isometry.

Let $\phi_k\colon (D_k, d_{D_k}) \to (\mathrm{hull}(D_k), d_{\mathrm{hull}(D_k)})$ be a bilipschitz map as in Lemma 5.19 with the property that, for each child d of D_k, ϕ_k is an isometry from J'_d to J_d.

Let ψ_k be the composition of $\phi_k \circ \varphi_k$ and $P_{k+1} = P_k \cup \mathrm{hull}(D_k)$. To obtain $\mathscr{J}_{k+1}$, we remove the joins $\mathscr{J}_k(D_k)$ from $\mathscr{J}_k$ and add the joins J_d where d is a child of D_k.

Since D_k has at most 3^n children when $n > 3$ and at most 16 children when $n = 3$, this concludes the induction step and the proof. □

5.3. **Proof of Proposition 5.1.** We construct now a sequence $(\mathbf{\Omega}_m)$ satisfying properties (1)–(4) in Proposition 5.1. Armed with Proposition 5.11, it suffices to find a sequence with the stability condition (1) and to verify condition (2).

5.3.1. *The initial step; step* 0. We begin with the n-cubes

$$\Omega_1 = [0,3]^n,\ \Omega_2 = [0,3]^{n-1} \times [-3,0],\ \text{and}\ \Omega_3 = [3,6] \times [0,3]^{n-1}$$

of side length 3. We denote

$$\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$$

and $\Omega = |\mathbf{\Omega}| = \Omega_1 \cup \Omega_2 \cup \Omega_3$; see Figure 36.

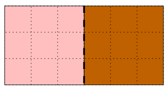

FIGURE 36. Faces of Ω_1 in $\partial_\cup \mathbf{\Omega}$

For consistency, we also denote

$$\mathbf{\Omega}_0 = (\Omega_{0,1}, \Omega_{0,2}, \Omega_{0,3}) = (\Omega_1, \Omega_2, \Omega_3).$$

Clearly $\mathbf{\Omega}_0$ satisfies the tripod property. Moreover, $\Omega_{0,1}$ is the only $\mathcal{C}$-cube in $\mathbf{\Omega}_0$, and $\partial_\cup \mathbf{\Omega}_0 \subset |\mathcal{C}(\mathbf{\Omega}_0)|$.

5.3.2. *The first step; a $\mathcal{C}$-modification.* First scale $\mathbf{\Omega}_0$ by 3, and denote

$$\mathbf{\Omega}'_1 = 3\mathbf{\Omega}_0 = (3\Omega_{0,1}, 3\Omega_{0,2}, 3\Omega_{0,3}).$$

We apply Lemma 5.8 to $C = 3\Omega_{0,1}$ and obtain an essential partition

$$\mathbf{\Omega}_1 = (\Omega_{1,1}, \Omega_{1,2}, \Omega_{1,3}) = (3\Omega_{0,1} - (A_2 \cup A_3), 3\Omega_{0,2} \cup A_2, 3\Omega_{0,3} \cup A_3)$$

of $|\mathbf{\Omega}'_1|$ into n-cells satisfying the tripod property, where A_2 and A_3 are atoms from the process of Lemma 4.25, see Figure 37.

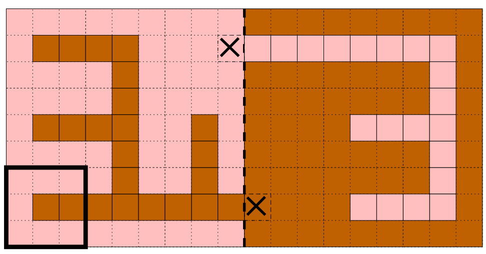

FIGURE 37. An example of the evolution of $\partial_\cup \mathbf{\Omega}_1$ with cube $[0,3]^3$ emphasized.

As a result, $\mathcal{C}(\mathbf{\Omega}_1) = \varnothing$ and $\partial_\cup \mathbf{\Omega}_1 \subset |\mathcal{D}(\mathbf{\Omega}_1; \mathbf{\Omega}'_1)|$. In particular, $\mathbf{\Omega}_1$ has the tripod property and the pair $\mathbf{\Omega}_0$ and $\mathbf{\Omega}_1$ satisfies assertion (1)-(3).

In the proof of Lemma 5.8, we are free to use any maximal forest Σ. In particular, we may assume that

(I_1) $[0,3]^n$ is a leaf of Σ.

Then $[0,3]^n \cap \Omega_{1,3}$ is a building block with two unit cubes; for an example, see Figure 37.

5.3.3. *The second step; an $\mathcal{E}$-modification.* Having essential partitions $\mathbf{\Omega}_0$ and $\mathbf{\Omega}_1$ at our disposal, Proposition 5.11 will produce the remaining $(\mathbf{\Omega}_m)$. However, to satisfy the stability condition of Proposition 5.1, the construction of $\mathbf{\Omega}_2$ is considered separately and then the general case of $\mathbf{\Omega}_m$ for $m \geq 3$.

Since $\mathbf{\Omega}_1$ is obtained by a $\mathcal{C}$-modification from $\mathbf{\Omega}_0$, an $\mathcal{E}$-modification (Lemma 5.10) will yield $\mathbf{\Omega}_2$; see Figure 38.

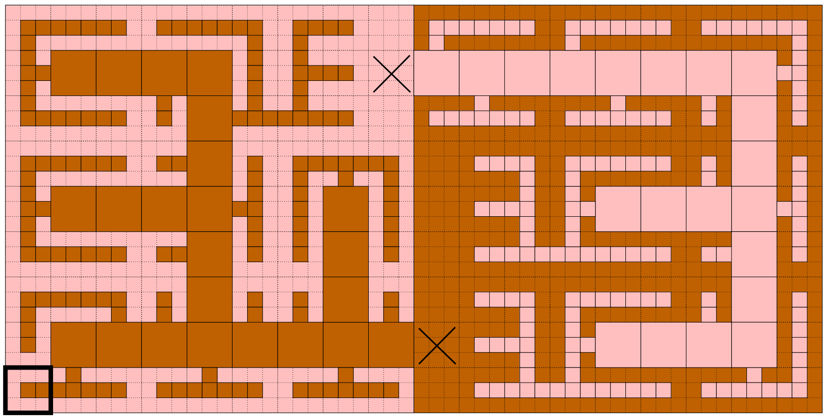

FIGURE 38. An example of $\mathbf{\Omega}_2$ with cube $[0,3]^3$ highlighted.

Since $D_0 = [0,3]^n \in \mathcal{D}(\mathbf{\Omega}_1; \mathbf{\Omega}_0)$, we satisfy (1) of Proposition 5.1 with the additional condition

(I_2) $\mathbf{\Omega}_2 \cap [0,3]^n = \mathbf{\Omega}_1 \cap [0,3]^n$

on $\mathbf{\Omega}_2$, as in Figure 38.

5.3.4. *Third and general induction step.* Given essential partitions $\mathbf{V}_i = \mathbf{\Omega}_i$, for $i = 0,1,2$ satisfying tripod property and (1)-(3) as in Proposition 5.11, this proposition yields a sequence of essential partitions $\mathbf{\Omega}_m$ for $m \geq 3$ satisfying the tripod property and conditions (a)-(f). In particular,

$$\mathbf{\Omega}_m \cap [0, 3^{m-1}]^n = \mathbf{\Omega}_{m-1} \cap [0, 3^{m-1}]^n.$$

for $m \geq 3$.

The specific initial choices show that the sequence $(\mathbf{\Omega}_m)$ satisfies (1) in Proposition 5.1. Since $(\mathbf{\Omega}_m)$ satisfies the conditions of Proposition 5.11, properties (2) and (3) of the claim follow directly.

We conclude the proof of Proposition 5.1 by showing that, for $p = 1,2,3$, $\Omega_p = \bigcup_{m\geq 0} \Omega_{m,p}$ is a bilipschitz equivalent to $\mathbb{R}^{n-1} \times [0,\infty)$ in its inner geometry.

By (2b), (Ω_p, d_{Ω_p}) is bilipschitz equivalent to $(\mathrm{hull}(\Omega_p), d_{\mathrm{hull}(\Omega_p)})$ for each p. Since $\mathrm{hull}(\Omega_3)$ is a monotone union of (ν,λ)-molecules, where ν and λ depend only on n, (Ω_3, d_{Ω_3}) is bilipschitz equivalent to $\mathbb{R}^{n-1} \times [0,\infty)$ by Proposition 3.8.

Concerning $\mathrm{hull}(\Omega_2)$, we observe first that $\mathrm{hull}(\Omega_2) \cap [0,\infty)^{n-1} \times [0,\infty)$ consists of an infinite collection of pair-wise disjoint (ν,δ)-molecules. Thus $(\mathrm{hull}(\Omega_2), d_{\mathrm{hull}(\Omega_2)})$ is bilipschitz equivalent to $[0,\infty)^{n-1} \times (-\infty, 0]$ as we may apply Proposition 3.5 to these molecules separately. Since components of $\mathrm{hull}(\Omega_2) \cap [0,\infty)^{n-1} \times [0,\infty)$ do not

meet $\partial[0,\infty)^{n-2}\times\mathbb{R}$, we obtain a bilipschitz homeomorphism $[0,\infty)^{n-1}\times(-\infty,0]\to(\mathrm{hull}(\Omega_2), d_{\mathrm{hull}(\Omega_2)})$ which is the identity on the boundary $\partial[0,\infty)^{n-1}\times(-\infty,0]$.

We are left with $\mathrm{hull}(\Omega_1)$. Since $\mathrm{hull}(\Omega_{1,m}) = [0,3^{m+1}]^n$ for every $m\geq 1$, $\mathrm{hull}(\Omega_1) = [0,\infty)^n$.

This completes the construction of a rough Rickman partition of $[0,\infty)^{n-1}\times\mathbb{R}$ and the proof of Proposition 5.1. $\square$

6. From cubes to simplices

In this section we introduce a particular triangulation of the pair-wise common boundary $\partial_\cup\mathbf{\Omega}$ of a rough Rickman partition $\mathbf{\Omega} = (\Omega_1,\Omega_2,\Omega_3)$. While the construction of the domains Ω_p is facilitated by using cubes as fundamental units, an Alexander-type mapping is more naturally described using simplices. We wish to remind the reader that the rough Rickman partition $\mathbf{\Omega}$ must be modified once more to obtain a Rickman partition $\widetilde{\mathbf{\Omega}}$ supporting a suitable BLD-mapping on $\partial_\cup\widetilde{\mathbf{\Omega}}$. The triangulation of $\partial_\cup\mathbf{\Omega}$ and a parity function carried by it have important roles in the construction of $\widetilde{\mathbf{\Omega}}$ in the next section.

The space $\mathbb{R}^n$ has a natural structure as a CW-complex with unit cubes $[0,1]^n+v$, $v\in\mathbb{Z}^n$, as n-cells, and the k-dimensional faces of these cubes as k-cells. Every $(n-1)$-cube Q of this complex has a natural subdivision into $(n-1)$-simplices. In what follows the convex hull of points $v_0,\ldots,v_k$ in $\mathbb{R}^k$, $0\leq k\leq n-1$, is

$$[v_0,\ldots,v_k].$$

Let Q be an $(n-1)$-cube in $\mathbb{R}^n$ and, for $k=0,\ldots,n-1$, Q_k a k-dimensional face of Q. The n-tuple $\mathcal{Q}=(Q_0,\ldots,Q_{n-1})$ a *flag in* Q if

$$Q_0\subset Q_1\subset\cdots\subset Q_{n-1}=Q. \tag{6.1}$$

Each k-cell Q_k has a uniquely defined barycenter c_{Q_k} and, by (6.1), the vectors $c_{Q_0}-c_{Q_{n-1}},\ldots,c_{Q_{n-2}}-c_{Q_{n-1}}$ are linearly independent with

$$S_{\mathcal{Q}} = [c_{Q_0},\ldots,c_{Q_{n-1}}],$$

an n-simplex contained in Q. We say that $S_{\mathcal{Q}}$ is the *simplex induced by the flag* $\mathcal{Q}$. Furthermore,

$$Q=\bigcup_{\mathcal{Q}} S_{\mathcal{Q}},$$

the union over all flags $(Q_0,\ldots,Q_n)$ in Q. Two $(n-1)$-simplices $S_{\mathcal{Q}}$ and $S_{\mathcal{Q}'}$, determined by different flags $\mathcal{Q}$ and $\mathcal{Q}'$, may intersect but they have no common interior. Thus simplices induced by flags triangulate Q.

Since simplices induced by flags are determined by the barycenters of lower-dimensional faces of $(n-1)$-cubes, every $(n-1)$-dimensional subcomplex $\mathbb{X}$ of $\mathbb{R}^{n-1}$, which is a union of its $(n-1)$-cells, admits a triangulation with simplices induced by flags. We call the simplicial complex associated to such triangulation the *standard simplicial structure of* $\mathbb{X}$. Note that since simplices in the standard simplicial structure arise as a subdivision of unit cubes in $\mathbb{R}^n$, the k-simplices, for $0<k\leq n$, in the standard simplicial structure have diameter between $1/2$ and $\sqrt{n}/2$.

Convention. *From now on we tacitly assume that a given $(n-1)$-simplex σ in an $(n-1)$-dimensional cubical complex $\mathbb{X}$ has the standard simplicial structure of $\mathbb{X}$.*

In particular, the pair-wise common boundary $\partial_\cup\mathbf{\Omega}$ of a Rickman partition $\mathbf{\Omega}$ admits this standard simplicial structure.

There is an elementary labeling function associated to the standard simplicial structure. Let $\mathbb{X}$ be an $(n-1)$-dimensional subcomplex of $\mathbb{R}^n$ so that $\mathbb{X}$ is a union of its $(n-1)$-cells and let $\mathbb{X}^{(0)}$ be the vertices of the standard simplicial structure.

Since every vertex v in $\mathbb{X}$ is a barycenter of a unique unit cube Q_v of maximal dimension in the cubical complex $\mathbb{X}$ (cf. (6.1)), the map

$$\vartheta_{\mathbb{X}} \colon \mathbb{X}^{(0)} \to \{0,\dots,n-1\},\ v \mapsto \dim Q_v,$$

is well-defined. Moreover, $\vartheta_{\mathbb{X}}(\sigma) = \{0,\dots,n-1\}$ for every $(n-1)$-simplex σ in the standard simplicial structure of $\mathbb{X}$. We call $\vartheta_{\mathbb{X}}$ the *labeling function of* $\mathbb{X}$.

6.1. **Parity functions.** Let $\mathbf{\Omega} = (\Omega_1,\Omega_2,\Omega_3)$ be a rough Rickman partition of $\mathbb{R}^n$ and let σ be an $(n-1)$-simplex in $(\partial_\cup \mathbf{\Omega})^{(n-1)}$. Then $\sigma = [v_0,\dots,v_{n-1}]$, where $0 \le k \le n-1$ and v_k is a barycenter of a k-cube in $\partial_\cup \mathbf{\Omega}$. Since $\partial_\cup \mathbf{\Omega}$ is the pair-wise common boundary, σ lies on the boundary of exactly two domains in $\mathbf{\Omega}$. We say that σ is $\mathbf{\Omega}$*-positive* if there exist i and j with $\sigma \subset \Omega_i \cap \Omega_j$ and

(1) $j = i+1 \mod 3$, and
(2) there exists a vector $v \in \mathbb{R}^n$ with $v_{n-1} + v \in \Omega_i$ and

$$\det((v_0 - v_{n-1}),\dots,(v_{n-2} - v_{n-1}), v) > 0. \tag{6.2}$$

Otherwise, σ is $\mathbf{\Omega}$*-negative.* A vector v satisfying (6.2) is called an *oriented normal of* σ if v is orthogonal to $v_k - v_{n-1}$ for every $0 \le k \le n-1$.

The *parity function of* $\mathbf{\Omega}$ *is the function* $\nu_{\mathbf{\Omega}} \colon (\partial_\cup \mathbf{\Omega})^{(n-1)} \to \{\pm 1\}$ defined by

$$\nu_{\mathbf{\Omega}}(\sigma) = \begin{cases} 1, & \sigma \text{ is } \mathbf{\Omega}\text{−positive}, \\ -1, & \sigma \text{ is } \mathbf{\Omega}\text{−negative}. \end{cases}$$

The next lemma describes the change of the parity on adjacent simplices.

Lemma 6.1. *Let $\mathbf{\Omega} = (\Omega_1,\Omega_2,\Omega_3)$ be a rough Rickman partition of $\mathbb{R}^n$. Suppose σ and σ' are $(n-1)$-simplices in $\partial\Omega_i$ sharing a common $(n-2)$-simplex. Then $\nu_{\mathbf{\Omega}}(\sigma) = -\nu_{\mathbf{\Omega}}(\sigma')$ if there exists $j \ne i$ so that $\sigma \cup \sigma' \subset \partial\Omega_j$, and $\nu_{\mathbf{\Omega}}(\sigma) = \nu_{\mathbf{\Omega}}(\sigma')$ otherwise.*

Proof. Let $\sigma = [v_0,\dots,v_{n-1}]$ and $\sigma' = [v'_0,\dots,v'_{n-1}]$. Suppose first that σ and σ' are contained in an $(n-1)$-dimensional plane P. We claim that

$$(v'_0 - v'_{n-1}) \wedge \cdots \wedge (v'_{n-2} - v'_{n-1}) = -(v_0 - v_{n-1}) \wedge \cdots \wedge (v_{n-2} - v_{n-1}). \tag{6.3}$$

It is then easy to verify the claim of the lemma as the oriented normal vectors of σ and σ' will be opposite normals of P.

Let $\mathcal{Q} = (Q_0,\dots,Q_n)$ and $\mathcal{Q}' = (Q'_0,\dots,Q'_n)$ be flags defining $\sigma = S_{\mathcal{Q}}$ and $\sigma' = S_{\mathcal{Q}'}$ respectively. Since σ and σ' have a common side, there exists $0 \le k \le n-1$ so that $v_i = v'_i$ for $i \ne k$.

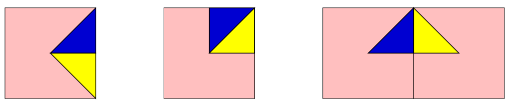

Figure 39. Congruence classes of planar $\sigma \cup \sigma'$ for $n = 3$ and $k = 0,1,2$.

Suppose first that $0 < k < n-1$. Then Q'_k and Q_k have a common face Q_{k-1} and are contained in Q_{k+1}. Since

$$c_{Q_{k-1}} - c_{Q_{k+1}} = (c_{Q_{k-1}} - c_{Q'_k}) + (c_{Q'_k} - c_{Q_{k+1}}) = (c_{Q_k} - c_{Q_{k+1}}) + (c_{Q'_k} - c_{Q_{k+1}}),$$

it follows that

$$\begin{aligned} v'_k - v_{n-1} &= v'_k - v_{k+1} + (v_{k+1} - v_{n-1}) \\ &= v_{k-1} - v_{k+1} - (v_k - v_{k+1}) + (v_{k+1} - v_{n-1}) \\ &= -(v_k - v_{n-1}) + (v_{k-1} - v_{n-1}) + (v_{k+1} - v_{n-1}), \end{aligned}$$

and so

$$\begin{aligned}&(v_0' - v_{n-1}') \wedge \cdots \wedge (v_k' - v_{n-1}') \wedge \cdots \wedge (v_{n-2}' - v_{n-1}')\\ &\quad = (v_0 - v_{n-1}) \wedge \cdots \wedge (v_k' - v_{n-1}) \wedge \cdots \wedge (v_{n-2} - v_{n-1})\\ &\quad = -(v_0 - v_{n-1}) \wedge \cdots \wedge (v_k - v_n) \wedge \cdots \wedge (v_{n-2} - v_{n-1}) :\end{aligned}$$

(6.3) holds. The cases $k = 0$ and $k = n - 1$ are similar.

Suppose now that σ and σ' are not contained in an $(n - 1)$-dimensional hyperplane. In this case, using the notation above, $v_{n-1}' \neq v_{n-1}$ and $v_k' = v_k$ for $0 \le k < n-1$. By construction of $\mathbf{\Omega}$, there also exists an n-cube Q having σ and σ' on its boundary. In particular, $w = c_Q - v_{n-1}$ and $w' = c_Q - v_{n-1}'$ are orthogonal to σ and σ', respectively.

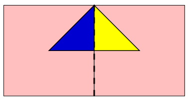

FIGURE 40. Fold-out of the congruence class of $\sigma \cup \sigma'$ for $n = 3$.

Since the n-simplices $[v_0, \ldots, v_{n-1}, c_Q]$ and $[v_0', \ldots, v_{n-1}', c_Q]$ are planar in $\mathbb{R}^{n+1}$ and share an $(n - 1)$-dimensional face, we have, by the previous argument,

$$(v_0' - c_Q) \wedge \cdots \wedge (v_{n-1}' - c_Q) = -(v_0 - c_Q) \wedge \cdots \wedge (v_{n-1} - c_Q),$$

so that

$$(v_0' - v_{n-1}') \wedge \cdots \wedge (v_{n-2}' - v_{n-1}') \wedge w' = -(v_0 - v_{n-1}) \wedge \cdots \wedge (v_{n-2} - v_{n-1}) \wedge w.$$

Since Q is contained in one of the elements of the partition $\mathbf{\Omega}$, the claim now follows by considering separately cases $Q \subset Q_i$ and $Q \subset Q_j$, where $j = i + 1 \mod 3$; in both cases the oriented normals for σ and σ' are either w and $-w'$, or $-w$ and w', respectively. □

7. Pillows and pillow covers

In this section we establish the most significant case, $p = 3$, of Proposition 1.5. The material in this section is almost verbatim to [13, Section 7].

Proposition 7.1. *Let $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ be a rough Rickman partition of $\mathbb{R}^n$ supporting the tripod property. Then there exists a Rickman partition $\widetilde{\mathbf{\Omega}} = (\widetilde{\Omega}_1, \widetilde{\Omega}_2, \widetilde{\Omega}_3)$ of $\mathbb{R}^n$ for which the Hausdorff distance of $\partial_\cup \mathbf{\Omega}$ and $\partial_\cup \widetilde{\mathbf{\Omega}}$ is at most 1.*

The proof of Proposition 1.5 is based on a construction of what we call a pillow cover of $\partial_\cup \mathbf{\Omega}$, and yields the final essential partition $\widetilde{\mathbf{\Omega}}$. The labeling and parity functions of $\mathbf{\Omega}$ lead at once to a BLD-map $\partial_\cup \widetilde{\mathbf{\Omega}} \to \widehat{\mathbb{S}}^{n-1}$, where $\widehat{\mathbb{S}}^{n-1} = \mathbb{S}^{n-1} \cup \mathbb{B}^{n-1}$. The bound on the Hausdorff distances of $\partial_\cup \mathbf{\Omega}$ and $\partial_\cup \widetilde{\mathbf{\Omega}}$ is immediate from the pillow construction.

We discuss first the pillow construction locally for planar $(n-1)$-cells contained in $\partial_\cup \mathbf{\Omega}$. For notational convenience let $E \subset \partial_\cup \mathbf{\Omega}$ be an $(n - 1)$-cell contained in a hyperplane P of $\mathbb{R}^n$ so that $E \subset \Omega_i \cap \Omega_j$ for some $i \neq j$. Throughout Sections 7.1–7.4 we consider E fixed but arbitrary. We may take $P = \mathbb{R}^{n-1}$. Then E inherits a standard simplicial structure from $\partial_\cup \mathbf{\Omega}$. We denote by $\nu = \nu_{E,\mathbf{\Omega}} \colon E^{(n-1)} \to \{\pm 1\}$ the restriction of the parity function ν_Ω to E. Similarly, $\vartheta = \vartheta_{E,\mathbf{\Omega}} \colon E^{(0)} \to \{0, \ldots, n - 1\}$ is the restriction of the labeling function $\vartheta_{\partial_\cup \mathbf{\Omega}}$ to E.

Let $\mathcal{E}$ be the adjacency graph $\Gamma(E^{(n-1)})$ and fix a maximal tree $\widehat{\mathcal{E}}$ in $\mathcal{E}$. Contrary to the case of maximal trees of adjacency graphs of cubical complexes, we consider

$\hat{\mathcal{E}}$ as a directed tree, and fix orientation on $\hat{\mathcal{E}}$ so that $\hat{\mathcal{E}}$ is connected and all simplices in $\hat{\mathcal{E}}$ have at most one outgoing edge and (possibly several or no) incoming edges.

Suppose σ is an $(n-1)$-simplex σ of $\hat{\mathcal{E}}$ and the $(n-2)$-simplex τ is a face of σ. Let σ' be an $(n-1)$-simplex in $\hat{\mathcal{E}}$ adjacent to σ: $\sigma' \cap \sigma = \tau$. Then τ is an *entry face of* σ if the edge between σ and σ' is an incoming edge to σ, and τ is an *exit face of* σ if it is the (unique) outgoing edge from σ. If τ is an entry or an exit face of a simplex, τ is considered *open*, otherwise τ is a *closed face of* σ; see Figure 41.

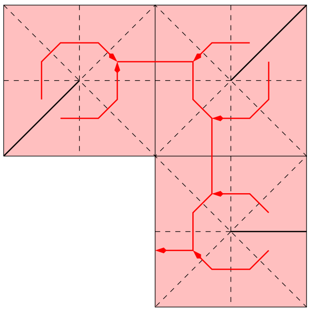

FIGURE 41.

7.1. **Pillow of a simplex.** As a preparatory step, let $\tau = [v_1, \ldots, v_{n-1}]$ be an $(n-2)$-simplex in $\mathbb{R}^{n-1}$, and consider τ as a face of an $(n-1)$-simplex σ in E. We define a subdivision $\tau_0, \ldots, \tau_{n-1}$ of τ as follows. For $i = 1, \ldots, n-1$, let

$$\tau_i = [(v_1 + v_i)/2, \ldots, (v_{n-1} + v_i)/2] \subset \tau.$$

Then τ_i is an $(n-2)$-simplex congruent to τ having diameter $(\operatorname{diam} \tau)/2$ and having v_i as a vertex; see Figure 42. Let $\tau_0 = \tau - \bigcup_{i=1}^{n-1} \tau_i$. Note that when $n = 3, 4$, τ_0 is an $(n-2)$-simplex whereas τ_0 is a more general simplicial set for $n > 4$.

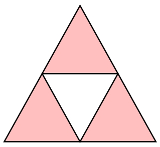

FIGURE 42. 2-simplices τ_1, τ_2, τ_3 surrounding τ_0 in a subdivision of τ; $n = 4$.

Definition 7.2. *Let $u\colon \tau \to [-1, 1]$ be a continuous function on τ. Then u is an* opening *if $u|\operatorname{int} \tau_0 > 0$ and $u|\tau \setminus \operatorname{int} \tau_0 = 0$. Similarly, u is a* shuffle *if*

(1) *$u|\operatorname{int} \tau_0 > 0$,*
(2) *there exist $i \neq j$ in $\{1, \ldots, n-1\}$ so that $u|\operatorname{int} \tau_i > 0$ and $u|\operatorname{int} \tau_j < 0$, and*
(3) *$u|\tau \setminus (\operatorname{int} \tau_0 \cup \operatorname{int} \tau_i \cup \operatorname{int} \tau_j) = 0$.*

Remark 7.3. *Note that, if $u\colon \tau \to [-1, 1]$ is either an opening or a shuffle, $u|\partial\tau = 0$.*

To each $(n-1)$-simplex σ in E, we set

$$\ell_\sigma = \begin{cases} 2, & \nu(\sigma) = -1, \\ 4, & \nu(\sigma) = 1, \end{cases}$$

and introduce a family of functions

$$\Psi_\sigma\colon \sigma \times \{1, \ldots, \ell_\sigma\} \to [-1, 1].$$

When $\nu(\sigma) = -1$, $u_\sigma \colon \partial\sigma \to [-1,1]$ is defined as follows. Given a face τ of σ, we set $u_\sigma|\tau$ to be an opening if τ is either an entry or an exit face of σ. If τ is closed, $u_\sigma|\tau$ is the zero function. Thus we may fix $\Psi_\sigma \colon \sigma \times \{1,2\} \to [-1,1]$ satisfying

(1) $\Psi_\sigma(x,1) = 0$ and $\Psi_\sigma(x,2) = u_\sigma(x)$ for all $x \in \partial\sigma$, and
(2) $\Psi_\sigma(x,1) < 0 < \Psi_\sigma(x,2)$ for all $x \in \operatorname{int}\sigma$.

When $\nu_\sigma(\sigma) = 1$, two functions u_σ and v_σ on $\partial\sigma$ will be used in a similar way. Given a face τ of σ, take $u_\sigma|\tau$ to be an opening if τ is either an entry or an exit face of σ, and $u_\sigma|\tau = 0$, otherwise. As for v_σ, define $v_\sigma|\tau = 0$ for every face τ of σ which is not the exit face, and take v_σ to be a shuffle on the exit face of σ if such exists. Note that u_σ and v_σ have (essentially) pair-wise disjoint supports.

We may now fix a function $\Psi_\sigma \colon \sigma \times \{1,\ldots,4\} \to [-1,1]$ so that, for $x \in \partial\sigma$,

(1) $\Psi_\sigma(x,1) = \Psi_\sigma(x,2) = 0$ and $\Psi_\sigma(x,3) = \Psi_\sigma(x,4) = u_\sigma(x)$ if $v_\sigma(x) = 0$,

(2) $\Psi_\sigma(x,1) = \Psi_\sigma(x,2) = \Psi_\sigma(x,3) = v_\sigma(x)$ and $\Psi_\sigma(x,4) = 0$ if $v_\sigma(x) < 0$,

(3) $\Psi_\sigma(x,1) = 0$ and $\Psi_\sigma(x,2) = \Psi_\sigma(x,3) = \Psi_\sigma(x,4) = v_\sigma(x)$ if $v_\sigma(x) > 0$,

while for $x \in \operatorname{int}\sigma$,

(4) $\Psi_\sigma(x,1) < \Psi_\sigma(x,2) < \Psi_\sigma(x,3) < \Psi_\sigma(x,4)$ and
(5) $\Psi_\sigma(x,1) < 0 < \Psi_\sigma(x,4)$.

We may also assume that, for both parities $\nu(\sigma)$ and every σ, the function Ψ_σ satisfies the additional regularity condition

$$\Psi_\sigma(x,i+1) - \Psi_\sigma(x,i) \geq \operatorname{dist}(x,\partial\sigma)/10$$

for $x \in \sigma$ and $i \in \{1,\ldots,\ell_\sigma - 1\}$. Note that $1/2 \leq \operatorname{diam}\sigma < 1$, since σ belongs to the standard simplicial structure of $\mathbb{R}^n$.

The singular n-simplices

$$\hat\sigma_i = \{(x,\Psi_\sigma(x,i)) \colon x \in \sigma\}, \tag{7.1}$$

where $i \in \{1,\ldots,\ell_\sigma\}$, constitute the *sheets of* σ (as in [12]), and the union of sheets

$$\hat\sigma = \bigcup_i \hat\sigma_i \tag{7.2}$$

forms a *pillow cover on* σ. We say that $\hat\sigma$ is *L-Lipschitz* if Ψ_σ is L-Lipschitz.

Convention. *Since simplices σ are mutually congruent, we may assume, from now on, that mappings Ψ_σ are PL and uniformly Lipschitz, that is, there exists $L \geq 1$ so that every Ψ_σ is L-Lipschitz for every σ in $\partial_\cup\mathbf{\Omega}$ and, in particularly, in the cell E.*

Remark 7.4. *Observe that $\{\hat\sigma_1,\ldots\hat\sigma_{\ell_\sigma}\}$ is a (singular) triangulation of $\hat\sigma$ by singular n-simplices. This triangulation, however, does not induce a simplicial complex, since pair-wise intersections of these simplices are generally not unions of sides. For example, $\hat\sigma_1 \cap \hat\sigma_{\ell_\sigma}$ is not a union of faces of $\hat\sigma_1$.*

We consider next, in more detail, the complementary domains of $\hat\sigma$ in $\sigma\times\mathbb{R}$. Let

$$P_\sigma = \{(x,t) \in \sigma\times\mathbb{R} \colon \Psi_\sigma(x,1) \leq t \leq \Psi_\sigma(x,\ell_\sigma)\}.$$

We call P_σ a “pillow”. Let also

$$U_\sigma = \{(x,t) \in \sigma\times\mathbb{R} \colon t \geq \Psi_\sigma(x,\ell_\sigma)\}$$

and

$$L_\sigma = \{(x,t) \in \sigma\times\mathbb{R} \colon t \leq \Psi_\sigma(x,1))\}.$$

Independent of the parity of σ, U_σ and L_σ are bilipschitz equivalent to $\sigma\times[0,\infty)$ and $\sigma\times(-\infty,0]$, respectively. For example, for U_σ, there is the bilipschitz map

$$(x,t)\mapsto\begin{cases}(x,2(t-\Psi_\sigma(x,\ell_\sigma)), & \Psi_\sigma(x,\ell_\sigma)\le t\le 2\Psi_\sigma(x,\ell_\sigma)\\ (x,t), & t\ge 2\Psi_\sigma(x,\ell_\sigma)\end{cases}$$

and similarly for L_σ the map

$$(x,t)\mapsto\begin{cases}(x,2(t-\Psi_\sigma(x,1)), & \Psi_\sigma(x,1)\ge t\ge 2\Psi_\sigma(x,1)\\ (x,t), & t\le 2\Psi_\sigma(x,1)\end{cases}$$

Since $|\Psi_\sigma|\le 1$, these homeomorphism are the identity outside $\sigma\times[-2,2]$, and the bilipschitz constant of these homeomorphisms depends only on n and the Lipschitz constant of Ψ_σ. Similarly, P_σ is bilipschitz to an n-cell independent of the parity of σ.

For $\nu(\sigma)=-1$, we observe that ∂P_σ is an essentially disjoint union of $\hat\sigma=\hat\sigma_1\cup\hat\sigma_2$ together with a union of $(n-1)$-cells in $\partial\sigma\times\mathbb{R}$; see Figure 43.

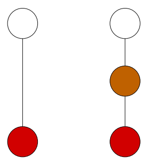

FIGURE 43. Adjacency graphs $\Gamma((\sigma\times\mathbb{R})\setminus\sigma)$ and $\Gamma((\sigma\times\mathbb{R})\setminus\hat\sigma)$ for σ with negative parity.

When $\nu(\sigma)=1$, the complementary domains have more complicated structure. Now $P_\sigma\setminus\hat\sigma$ has three components with closures P_σ^U, P_σ^M, and P_σ^L, respectively,

$$\begin{aligned}P_\sigma^U &= \{(x,t)\colon \Psi_\sigma(x,1)\le t\le\Psi_\sigma(x,2)\},\\ P_\sigma^M &= \{(x,t)\colon \Psi_\sigma(x,2)\le t\le\Psi_\sigma(x,3)\},\ \text{and}\\ P_\sigma^L &= \{(x,t)\colon \Psi_\sigma(x,3)\le t\le\Psi_\sigma(x,4)\};\end{aligned}$$

the letters 'U', 'M', and 'L' refer to 'upper', 'middle', and 'lower' domains in Lemma 7.6 below. Furthermore,

$$\hat\sigma\cap\partial P_\sigma^U=\sigma_1\cup\sigma_2,\ \hat\sigma\cap\partial P_\sigma^M=\sigma_2\cup\sigma_3,\ \text{and}\ \hat\sigma\cap\partial P_\sigma^L=\sigma_3\cup\sigma_4;$$

see Figure 44.

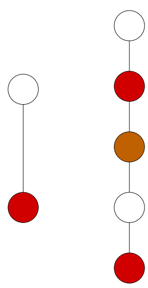

FIGURE 44. Adjacency graphs $\Gamma((\sigma\times\mathbb{R})\setminus\sigma)$ and $\Gamma((\sigma\times\mathbb{R})\setminus\hat\sigma)$ for σ with positive parity.

By definition of Φ_σ, both $\partial P_\sigma^U\setminus\hat\sigma$ and $\partial P_\sigma^L\setminus\hat\sigma$ consist of a single open $(n-1)$-cell contained in $\partial\sigma\times\mathbb{R}$. We denote these $(n-1)$-cells by D_σ^U and D_σ^L, respectively. In particular,

$$D_\sigma^U\subset\mathbb{R}^n\times(0,\infty)\ \text{and}\ D_\sigma^L\subset\mathbb{R}^n\times(-\infty,0).$$

7.2. **Pillow covers of adjacent simplices.** Recall that $E \subset \partial_{\cup}\mathbf{\Omega}$ is a planar $(n-1)$-cell, and, to simplify the notation, we have assumed $E \subset \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$.

Let σ be an $(n-1)$-simplex in E as before and suppose that σ' is another $(n-1)$-simplex in E sharing an $(n-2)$-simplex with σ. By changing the rôles of σ and $\hat\sigma$ if necessary, we may assume $\nu(\sigma') = -\nu(\sigma) = -1$.

Definition 7.5. *Pillows $\hat\sigma$ and $\hat\sigma'$ of σ and σ', respectively, are* compatible *if $\Psi_\sigma(\cdot,2) = \Psi_{\sigma'}(\cdot,1)$ and $\Psi_\sigma(\cdot,3) = \Psi_{\sigma'}(\cdot,2)$ on τ, where τ is the common face of σ and σ'.*

From now on we assume that $\hat\sigma$ and $\hat\sigma'$ are compatible pillows. The following lemma recapitulates Rickman's idea on using two types of pillows to permute the local rôles of the three domains.

Lemma 7.6. *Let $\hat\sigma$ and $\hat\sigma'$ be compatible pillows of σ and σ', respectively. Then*
$$((\sigma \cup \sigma') \times \mathbb{R}) \setminus (\hat\sigma \cup \hat\sigma')$$
has three components Ω^U, Ω^M, and Ω^L satisfying
$$\overline{\Omega^U} = U_\sigma \cup P_\sigma^L \cup U_{\sigma'},\ \overline{\Omega^M} = P_\sigma^M \cup P_{\sigma'},\ \text{and}\ \overline{\Omega^L} = L_\sigma \cup P_\sigma^U \cup L_{\sigma'}.$$

Proof. It suffices to observe that the closures of P_σ^L and $U_{\sigma'}$ meet in the $(n-1)$-cell
$$\{(x,t)\colon \tau \times \mathbb{R} \colon \Phi_\sigma(x,3) \le t \le \Phi_\sigma(x,4)\}.$$
Similarly, $P_\sigma^U \cap L_{\sigma'}$ is an $(n-1)$-cell. □

Using the notation of Lemma 7.6, we make now few observations on the natural triangulation of $\tilde\sigma \cup \tilde\sigma'$ into sheets and domains Ω^U, Ω^M, Ω^L.

For $\hat\sigma'$, the pair-wise intersections of domains Ω^L, Ω^M, Ω^U with $\hat\sigma' \times \mathbb{R}$ are (up to a closure) $L_{\sigma'}$, $P_{\sigma'}$, and $U_{\sigma'}$. Thus
$$\partial\Omega^L \cap \hat\sigma' = \hat\sigma'_1,\ \partial\Omega^M \cap \hat\sigma' = \hat\sigma'_1 \cup \hat\sigma'_2,\ \text{and}\ \partial\Omega^U \cap \hat\sigma' = \hat\sigma'_2.$$

The situation is slightly more complicated with $\hat\sigma$. Note first that $\Omega^M \cap (\sigma \times \mathbb{R})$ is P_σ^M up to closure. Thus
$$\partial\Omega^M \cap \hat\sigma = \hat\sigma_2 \cup \hat\sigma_3,$$
and we have
$$\hat\sigma_2 = \Omega^L \cap \Omega^M \cap (\sigma \times \mathbb{R})\ \text{and}\ \hat\sigma_3 = \Omega^U \cap \Omega^M \cap (\sigma \times \mathbb{R}).$$

Moreover,
$$\overline{\Omega^L \cap (\sigma \times \mathbb{R})} = \overline{L_\sigma \cup P_\sigma^U}\ \text{and}\ \overline{\Omega^U \cap (\sigma \times \mathbb{R})} = \overline{U_\sigma \cup P_\sigma^L},$$
$$\partial\Omega^L \cap \hat\sigma = \partial L_\sigma \cup \partial P_\sigma^U = \hat\sigma_1 \cup \hat\sigma_3 \cup \hat\sigma_4,$$
and
$$\partial\Omega^U \cap \hat\sigma = \partial U_\sigma \cup \partial P_\sigma^L = \hat\sigma_4 \cup \hat\sigma_1 \cup \hat\sigma_2.$$

This 'shuffle' will allow our domains $\{\Omega_\ell\}$ to connect near $\partial_{\cup}\mathbf{\Omega}$. The proof of following lemma is left to the interested reader; the situation is captured by the suggestive figure in [13, Fig 7.2] and Figure 45.

Lemma 7.7. *With the notation above, we have*
$$\begin{aligned}\hat\sigma_1 &= \Omega^U \cap \Omega^L \cap (\sigma \times \mathbb{R}),\ \hat\sigma_2 = \Omega^L \cap \Omega^M \cap (\sigma \times \mathbb{R}),\\ \hat\sigma_3 &= \Omega^M \cap \Omega^U \cap (\sigma \times \mathbb{R}),\ \text{and}\ \hat\sigma_4 = \Omega^U \cap \Omega^L \cap (\sigma \times \mathbb{R}).\end{aligned}$$
Furthermore,
$$\hat\sigma'_1 = \Omega^L \cap \Omega^M \cap (\sigma' \times \mathbb{R}),\ \text{and}\ \hat\sigma'_2 = \Omega^M \cap \Omega^U \cap (\sigma' \times \mathbb{R}).$$

Our discussion shows that the domains Ω^U, Ω^M, and Ω^L are bilipschitz equivalent to either $(\sigma \cup \sigma') \times (0,\infty)$, $(\sigma \cup \sigma') \times (-\infty, 0)$, or to $\mathbb{B}^n$. Hence

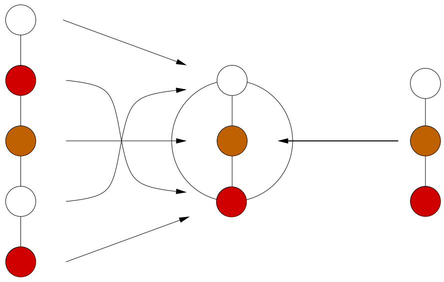

FIGURE 45. Adjacency graphs $\Gamma((\sigma\times\mathbb{R})\setminus\hat\sigma)$, $\Gamma(((\sigma\cup\sigma')\times\mathbb{R})\setminus(\hat\sigma\cup\hat\sigma'))$ and $\Gamma((\sigma'\times\mathbb{R})\setminus\hat\sigma')$ with maps of graphs induced by inclusions.

Lemma 7.8. *Let $\hat\sigma$ and $\hat\sigma'$ be compatible Lipschitz pillows on σ and σ', respectively. Then*

(1) *there exist bilipschitz homeomorphisms $h^U_{\sigma,\sigma'}\colon(\sigma\cup\sigma')\times(0,\infty)\to(\Omega^U,d_{\Omega^U})$ and $h^L_{\sigma,\sigma'}\colon(\sigma\cup\sigma')\times(-\infty,0)\to(\Omega^L,d_{\Omega^L})$ whose supports are contained in $\sigma\cup\sigma'\times[-1/2,1/2]$ so that $h^U_{\sigma,\sigma'}$ and $h^L_{\sigma,\sigma'}$ extend to BLD-maps $(\sigma\cup\sigma')\times[0,\infty)\to\overline{\Omega^U}$ and $(\sigma\cup\sigma')\times(-\infty,0]\to\overline{\Omega^L}$, respectively, and*
(2) *the closure of Ω^M is a bilipschitz n-cell.*

The bilipschitz (and BLD) constants are quantitative in the sense that they depend only on n, the Lipschitz constants of Ψ_σ and $\Psi_{\sigma'}$ and the minimal bilipschitz constants of homeomorphisms $\sigma\to\mathbb{B}^{n-1}$ and $\sigma'\to\mathbb{B}^{n-1}$.

7.3. **Maps on pairs of sheets.** The pillow construction on $\sigma\cup\sigma'$ gives rise to maps $\hat\sigma\cup\hat\sigma'\to\hat{\mathbb{S}}^{n-1}$. We discuss these local maps now in more detail.

Recall that $\hat{\mathbb{S}}^{n-1}=\mathbb{S}^{n-1}\cup\mathbb{B}^{n-1}\subset\mathbb{R}^n$ and write $\mathbb{S}^{n-1}=\mathbb{S}^{n-1}_+\cup\mathbb{S}^{n-1}_-$, where $\mathbb{S}^{n-1}_+$ and $\mathbb{S}^{n-1}_-$ are the upper and lower hemispheres of $\mathbb{S}^{n-1}$, i.e. $\mathbb{S}^{n-1}_+\cap\mathbb{S}^{n-1}_-=\partial\mathbb{B}^{n-1}$. Then $\mathbb{R}^n\setminus\hat{\mathbb{S}}^{n-1}$ has three components denoted D^U, D^L, and D^M so that $\partial D^U=\mathbb{S}^{n-1}_+\cup\mathbb{B}^{n-1}$, $\partial D^L=\mathbb{S}^{n-1}_-\cup\mathbb{B}^{n-1}$, and $\partial D^M=\mathbb{S}^{n-1}$. We fix n points $\{y_0,\ldots,y_{n-1}\}$ on $\partial\mathbb{B}^{n-1}$ and view $\hat{\mathbb{S}}^{n-1}$ as a CW-complex having three $(n-1)$-cells $\mathbb{S}^{n-1}_+$, $\mathbb{S}^{n-1}_-$, and $\mathbb{B}^{n-1}$ and vertices $\{y_0,\ldots,y_{n-1}\}$.

Let σ and σ' be adjacent $(n-1)$-simplices in E and let $\hat\sigma$ and $\hat\sigma'$ be compatible Lipschitz pillows on σ and σ', respectively. By changing the rôles of σ and σ' if necessary, we may assume $\nu(\sigma)=-\nu(\sigma')=1$. Let $\vartheta\colon(\sigma^{(0)}\cup\sigma'^{(0)})\to\{0,\ldots,n-1\}$ be the labeling function of $\mathbf{\Omega}$ restricted to $\sigma\cup\sigma'$.

Although the singular simplices $\Delta=\{\hat\sigma_1,\ldots,\hat\sigma_4,\hat\sigma'_1,\hat\sigma'_2\}$ again do not define a simplicial complex, there exists a continuous map $f\colon\hat\sigma\cup\hat\sigma'\to\hat{\mathbb{S}}^n$ satisfying

(S1) f maps each singular simplex in Δ to one of the simplices $\mathbb{S}^{n-1}_+$, $\mathbb{S}^{n-1}_-$, or $\mathbb{B}^{n-1}$ in a bilipschitz manner,
(S2) $f(v)=y_{\vartheta(v)}$ for all $v\in\sigma^{(0)}\cup(\sigma')^{(0)}$, and
(S3) if $\{X,Y\}\subset\{U,L,M\}$ is a pair then $f(\Omega^X\cap\Omega^Y)=D^X\cap D^Y$.

Since f is bilipschitz on singular simplices, it is discrete and

$$\frac{1}{\mathcal{L}}\ell(\gamma)\le\ell(f\circ\gamma)\le\mathcal{L}\ell(\gamma)$$

for all paths γ in $\sigma\cup\sigma'$, where $\mathcal{L}$ is the maximum of the bilipschitz constants of f restricted to simplices in Δ. Furthermore, in the sense of the following lemma, f is a branched cover in the interior of $\hat\sigma\cup\hat\sigma'$.

Lemma 7.9. *Let $O = (\hat\sigma \cup \hat\sigma') \cap \mathrm{int}\,(\sigma \cup \sigma') \times \mathbb{R}$. Then $f|O\colon O \to \hat{\mathbb{S}}^n$ is a branched cover and the branch set of $f|O$ is the set*

$$O \cap \{y \in \sigma \cap \sigma'\colon \Psi_\sigma(y,1) = \Psi_\sigma(y,4)\} \subset \mathbb{R}^n.$$

In particular, $f|O$ is an open map.

Proof. Let τ be the common face of σ and σ'. We denote $S = \hat\sigma \cup \hat\sigma'$ and

$$G = \bigcup_{i=1}^4 \mathrm{int}\,\hat\sigma_i \cup \bigcup_{j=1}^2 \mathrm{int}\,\hat\sigma'_j.$$

Then

$$S = G \cup (S \cap (\tau \times \mathbb{R})) \cup (S \cap \partial(\sigma \cup \sigma') \times \mathbb{R}).$$

Clearly $G \subset O$ and $f|G\colon G \to \hat{\mathbb{S}}^n$ is a local homeomorphism. Suppose now that $x = (y,t) \in O \cap (\tau \times \mathbb{R})$. Then $f(x) \in \mathbb{S}^n \cap \mathbb{B}^n$.

There are four cases to consider. Suppose first that y has a neighborhood O' so that $\Psi_\sigma(y',1) = \Psi_\sigma(y',2)$ for $y' \in O'$. Then also $\Psi_\sigma(y',1) = \Psi_{\sigma'}(y',1)$ and $\Psi_\sigma(y',3) = \Psi_\sigma(y',4) = \Psi_{\sigma'}(y',2)$ for $y' \in O'$ by compatibility, and so either $t = \Psi_\sigma(y,1) = \Psi_{\sigma'}(y,1)$ or $t = \Psi_\sigma(y,3) = \Psi_{\sigma'}(y,2)$. In either case, there are exactly three simplices T_U, T_L, T_M among the simplices $\{\hat\sigma_1,\ldots,\hat\sigma_4,\hat\sigma'_1,\hat\sigma'_2\}$ with $x \in T_U \cap T_L \cap T_M$ and $f(T_U) = \partial D^U$, $f(T_L) = \partial D^L$, $f(T_M) = \partial D^M$. When y has a neighborhood O' with $\Psi_\sigma(y',1) = \Psi_\sigma(y',3)$ or $\Psi_\sigma(y',2) = \Psi_\sigma(y',4)$ for $y' \in O'$, the argument is similar. In all these cases, f is a homeomorphism in a neighborhood of x.

In the remaining case, $x \in O \cap (\tau \times \mathbb{R})$ and $\Psi_\sigma(y,1) = \Psi_\sigma(y,4)$. Then x belongs to all six singular simplices, and f is a branched double cover near x. □

7.4. **Pillow covers of cells.** Suppose first that E is a planar $(n-1)$-cell, i.e. E is contained in an $(n-1)$-plane P. We may take $P = \mathbb{R}^{n-1}$ as in the beginning of Section 7.

Having $\nu = \nu_{E,\mathbf{\Omega}}$ and $\hat{\mathcal{E}} = \Gamma(E^{(n-1)})$ at our disposal, we fix, for every $\sigma \in E^{(n-1)}$, a pillow $\hat\sigma$ compatible with simplices adjacent to σ in E. The set

$$\hat E = \bigcup_{\sigma \in E^{(n-1)}} \hat\sigma$$

is called a *pillow on E*. By our convention, all pillows $\hat\sigma$ for $\sigma \in E^{(n-1)}$ are $\mathcal{L}$-Lipschitz for $\mathcal{L} \ge 1$, so that $\hat E$ is an *$\mathcal{L}$-Lipschitz pillow.*

Lemmas 7.6 and 7.8 on metric properties of the pillow construction for pairs of simplices have counterparts for planar n-cells. The proofs are verbatim so we merely state the results.

Lemma 7.10. *Let E be a cubical planar $(n-1)$-cell in $\mathbb{R}^{n-1}$ and $\hat E \subset E \times [-1/2,1/2]$ an $\mathcal{L}$-Lipschitz pillow on E. Then*

$$E \times [-1,1] \setminus \hat E$$

has three components Ω^U, Ω^M, and Ω^L, each bilipschitz equivalent to $\mathbb{B}^n$ in their inner metric, respectively, so that $\Omega^U \supset E \times \{1\}$ and $\Omega^L \supset E \times \{-1\}$. The bilipschitz constant is quantitative and depends only on n and $\mathcal{L}$.

Lemma 7.11. *Let E be a cubical planar $(n-1)$-cell in $\mathbb{R}^{n-1}$ and $\hat E \subset E \times [-1/2,1/2]$ an $\mathcal{L}$-Lipschitz pillow on E. Then*

(1) *there exists a bilipschitz homeomorphism $h^U_E\colon E \times (0,1) \to (\Omega^U, d_{\Omega^U})$ having a BLD-extension $\bar h^U_E\colon E \times [0,1] \to \overline{\Omega^U}$ so that $\bar h^U_E$ is the identity on $E \times \{1\} \cup \partial E \times [0,1]$.*

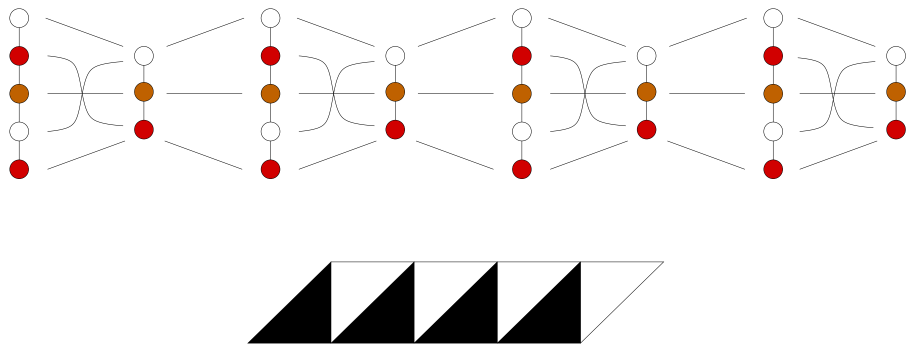

FIGURE 46. The adjacency of domains Ω^U, Ω^M, Ω^L over a 2-cell.

(2) *there exists a bilipschitz homeomorphism $h^L_E\colon E\times(-1,0)\to(\Omega^L,d_{\Omega^L})$ having a BLD-extension $\bar h^L_E\colon E\times[-1,0]\to\overline{\Omega^L}$ so that $\bar h^L_E$ the identity on $E\times\{-1\}\cup\partial E\times[-1,0]$.*

The statement is quantitative in the sense that the bilipschitz constant depends only on n and $\mathcal{L}$.

In order to define maps $\hat E\to\hat{\mathbb{S}}^n$, we fix points $\{y_0,\ldots,y_{n-1}\}\subset\mathbb{S}^{n-1}\cap\mathbb{B}^{n-1}$, as in Section 7.3. The following lemma is a counterpart of the construction in Section 7.3.

Lemma 7.12. *Let E be a cubical planar n-cell in $\mathbb{R}^n$ and $\hat E\subset E\times[-1/2,1/2]$ a pillow on E. Then there exists a map $f_E\colon\hat E\to\hat{\mathbb{S}}^n$, which is a branched cover on $\mathrm{int}\,\hat E=\hat E\cap(\mathrm{int}\,E\times\mathbb{R})$, so that $f_E|(\sigma\cup\sigma')$ satisfies (S1)-(S3) for every pair of adjacent simplices σ and σ' in $E^{(n-1)}$. The BLD-constant of $f_E|\mathrm{int}\,\hat E$ is quantitative in the sense that it depends only on n, $\mathcal{L}$, and points $\{y_0,\ldots,y_{n-1}\}$.*

Proof. It suffices to observe that f_E is readily obtained from the discussion in Section 7.3 and it suffices to discuss the uniformity of the BLD-constant of $f_E|\mathrm{int}\,\hat E$. Since E is given a standard simplicial structure, all simplices σ in $E^{(n-1)}$ are congruent. For every $\sigma\in E^{(n-1)}$ faces of σ are one of the three different types: *entry*, *exit*, and *closed* faces. By fixing opening and shuffle functions invariant under congruences, we may assume that pillows over simplices, with the same combinatorics, are congruent. More precisely, there exist simplices $\sigma_1,\ldots,\sigma_r$ in $E^{(n-1)}$ and compatible pillows, so that, for every $\sigma\in E^{(n-1)}$, there exists an isometry I_σ of $\mathbb{R}^n$, preserving $\mathbb{R}^{n-1}\times[0,\infty)$, and $1\le i_\sigma\le r$ so that $I_\sigma(\sigma)=\sigma_{i_\sigma}$ and $I_\sigma(\hat\sigma)=\hat\sigma_{i_\sigma}$.

Thus we fix a finite collection of Lipschitz maps $f_i\colon\hat\sigma_i\to\hat{\mathbb{S}}^{n-1}$ and use the isometries I_σ to obtain a map $f_E\colon\hat E\to\hat{\mathbb{S}}^{(n-1)}$. The BLD-constant of $f_E|\mathrm{int}\,\hat E$ then clearly depends only on the Lipschitz constants of this finite collection $f_1,\ldots,f_r$, depending only on n, $\mathcal{L}$, and the choice of points $\{y_0,\ldots,y_{n-1}\}$. $\square$

Remark 7.13. *The standard simplicial structure of E is not essential to the proof of Lemma 7.12. In fact, given any simplicial complex P in $\mathbb{R}^n$ with $|P|=E$, it is easy to observe that there exists a pillow $\hat E$ on E consisting of compatible pillows $\hat\sigma$ for $\sigma\in P^{(n-1)}$, and a map $f_{E,P}\colon\hat E\to\hat{\mathbb{S}}^{n-1}$ satisfying the properties of Lemma 7.12 with the only exception that the BLD-constant of $f_{E,P}|\mathrm{int}\,\hat E$ now depends also on the bilipschitz constants of affine parametrizations $[0,e_1,\ldots,e_{n-1}]\to\sigma$ for $\sigma\in P^{(n-1)}$. (Although, this observation is essential in what follows, we leave the simple modification of the proof of Lemma 7.12 to the interested reader.)*

Suppose now that E is a cubical $(n-1)$-cell in $\mathbb{R}^n$. Since E is a PL $(n-1)$-cell, there exists a PL homeomorphism $E\to E'$, where E' is an $(n-1)$-cell in $\mathbb{R}^{n-1}$.

More precisely, there exists a simplicial complex P so that $|P| = E$ and a simplicial homeomorphism $\varphi\colon E \to E'$ with respect to P.

Let E be a cubical $(n-1)$-cell E in $\mathbb{R}^n$ and let $\mathcal{Q}(E)$ be the collection of all unit n-cubes Q in $\mathbb{R}^n$ with $Q \cap \operatorname{int} E \neq \varnothing$, and $|\mathcal{Q}(E)|$ the union of these cubes. Set

$$\mathcal{N}(E) = B_\infty(E, 1/3) \cap |\mathcal{Q}(E)|.$$

In particular, we have

$$\mathcal{N}(E') = E' \times [-1, 1]$$

for a planar $(n-1)$-cell E' in $\mathbb{R}^n$. More generally, the pair $(\mathcal{N}(E), E)$ is PL-homeomorphic to proper cell pair $(\bar{B}^n, \bar{B}^{n-1})$; see [16, Chapter 4].

We apply these observations to small $(n-1)$-cells in $\mathbb{R}^n$, and summarize the needed properties in the following lemma, omitting details. Note that the uniform bound of the bilipschitz constant follows directly from the finiteness of congruence classes of $(n-1)$-cells in statement.

Lemma 7.14. *Let E be a cubical $(n-1)$-cell in a cube $Q \subset \mathbb{R}^n$ of side length 3. Then there exist $\mathcal{L} \geq 1$ depending only on n, a planar cubical $(n-1)$-cell E', and an $\mathcal{L}$-bilipschitz PL-homeomorphism $\varphi_E\colon \mathcal{N}(E) \to \mathcal{N}(E')$ so that $\varphi_E(E) = E'$. Moreover, there is a simplicial complex P so that $|P| = E$ and φ_E is piecewise affine with respect to P.*

Having Lemma 7.14 at our disposal, we may define pillow covers for small $(n-1)$-cells in $\mathbb{R}^n$. Let E be a cubical $(n-1)$-cell contained in a cube of side length 3. Suppose E' is a planar $(n-1)$-cell and $\varphi_E\colon \mathcal{N}(E) \to \mathcal{N}(E')$ a PL-homeomorphism as in Lemma 7.14. Then $\varphi_E(E^{(n-1)})$ is a triangulation of E'. Although $\varphi_E(E^{(n-1)})$ is not necessarily the standard triangulation of E', we obtain, by a simple modification to the pillow construction, a pillow $\widehat{E'}$ on E' in $\mathcal{N}(E')$ with respect to this triangulation, and call $\widehat{E} = \phi_E^{-1}(\widehat{E'})$ a pillow cover of E.

Given an $(n-1)$-simplex σ in E, we also say that $\hat{\sigma} = \varphi^{-1}(\widehat{E} \cap (\varphi(\sigma) \times [-1, 1]))$ is the *pillow over σ in $\widehat{E}$*. By finiteness of congruence classes, we conclude that the results in the beginning of this section hold also for these pillow covers almost verbatim.

7.5. **Proof of Proposition 7.1.** Let $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ be a rough Rickman partition of $\mathbb{R}^n$ having the tripod property. Thus $\partial_\cup \mathbf{\Omega}$ has an essential partition into cubical $(n-1)$-cells $\Delta = \{E_\ell\}_{\ell \geq 0}$.

Given adjacent E_ℓ and $E_{\ell'}$ in Δ belonging to different $\mathbf{\Omega}$-equivalence classes (recall Definition 4.2), there exists, by property $(\Delta 2)$ of Definition 4.4, a unique $E_{\ell''}$ in Δ so that the cells E_ℓ, $E_{\ell'}$, and $E_{\ell''}$ are mutually adjacent, contained in the same cube of side length 3, and belong to different $\mathbf{\Omega}$-equivalence classes. We denote $E_\ell \sim E_{\ell'}$. The relation $\sim$ defines an equivalence relation in Δ which subdivides Δ into equivalence classes containing exactly three elements.

Let

$$\mathcal{N}(\partial_\cup \mathbf{\Omega}) = B_\infty(\partial_\cup \mathbf{\Omega}, 1/3)$$

be the $(1/3)$-neighborhood of $\partial_\cup \mathbf{\Omega}$ in $\mathbb{R}^n$, and for each ℓ define

$$\mathcal{N}_\ell = \{x \in \mathcal{N}(\partial_\cup \mathbf{\Omega})\colon \operatorname{dist}_\infty(x, E_\ell) = \operatorname{dist}_\infty(x, \partial_\cup \mathbf{\Omega})\}.$$

Then $\{\mathcal{N}_\ell\}_{\ell \geq 0}$ is an essential partition of $\mathcal{N}(\partial_\cup \mathbf{\Omega})$. Moreover, $\mathcal{N}_\ell$ is PL-homeomorphic to $\mathcal{N}(E_\ell)$ for every ℓ. Due to finite number of congruence classes of $\mathcal{N}_\ell$ and $\mathcal{N}(E_\ell)$, we have that $\mathcal{N}_\ell$ is bilipschitz to $\mathcal{N}(E_\ell)$, the constant depending only on n.

Suppose E_{ℓ_0}, E_{ℓ_1}, and E_{ℓ_2} are equivalent $(n-1)$-cells in Δ. We create pillows $\hat{E}_{\ell_0}$, $\hat{E}_{\ell_1}$ and $\hat{E}_{\ell_2}$ simultaneously. Let $E_{[\ell]} = E_{\ell_0} \cup E_{\ell_1} \cup E_{\ell_2}$ and $\mathcal{N}_{[\ell]} = \mathcal{N}_{\ell_0} \cup \mathcal{N}_{\ell_1} \cup \mathcal{N}_{\ell_2}$.

We fix, for $m = 0, 1, 2$, indices $\{i_m, j_m, k_m\} = \{1, 2, 3\}$ so that $E_{\ell_m} \cap \Omega_{k_m}$ is an $(n-2)$-cell and $E_{\ell_m} \subset \Omega_{i_m} \cap \Omega_{j_m}$.

Let

$$Y = \left(\mathbb{R}^{n-1} \times \{0\}\right) \cup \left(\{0\} \times \mathbb{R}^{n-2} \times [0, \infty)\right) \subset \mathbb{R}^n.$$

Since $E_{[\ell]}$ is a union of three essentially disjoint $(n-1)$-cells meeting in an $(n-2)$-cell, there is a PL-embedding $\psi_{[\ell]} \colon E_{[\ell]} \to Y$ so that

$$E'_{\ell_0} \subset (-\infty, 0] \times \mathbb{R}^{n-2}, \ E'_{\ell_1} \subset [0, \infty) \times \mathbb{R}^{n-2}, \text{ and } E'_{\ell_2} \subset \{0\} \times \mathbb{R}^{n-2} \times [0, \infty),$$

where each $E'_{\ell_m} = \psi_{[\ell]}(E_{\ell_m})$ is a planar $(n-1)$-cell. Let also $E'_{[\ell]} = \psi_{[\ell]}(E_{[\ell]})$.

The map $\psi_{[\ell]}$ extends to a PL-homeomorphism $\psi_{[\ell]} \colon \mathcal{N}_{[\ell]} \to \mathcal{N}(E'_{[\ell]})$, where $\mathcal{N}(E'_{[\ell]}) = \bigcup_{m=0}^{2} \mathcal{N}(E'_{\ell_m})$. The connected components of $\mathcal{N}(E'_{[\ell]}) \setminus Y$ are $U_m = \psi_{[\ell]}(\operatorname{int} \Omega_m \cap \mathcal{N}_{[\ell]})$ for $m = 0, 1, 2$.

Again, by finiteness of congruence classes, $\psi_{[\ell]}$ is bilipschitz with constant depending only on n. Each homeomorphism ψ induces a triangulation $\psi_{[\ell]}(E^{(n-1)}_{[\ell]})$ on $E'_{[\ell]}$, and we denote by ν the parity function $\sigma \mapsto \nu_{\mathbf{\Omega}}(\psi^{-1}_{[\ell]} \circ \sigma)$ defined on simplices in $\psi_{[\ell]}(E^{(n-1)}_{[\ell]})$.

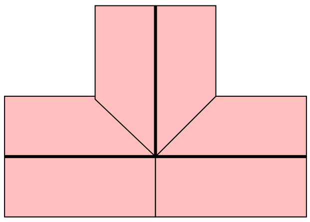

FIGURE 47. Cells E'_ℓ, $E'_{\ell'}$, and $E'_{\ell''}$ meeting at $\{0\} \times \mathbb{R}^{n-2} \times \{0\}$ and partition of $\mathcal{N}(E'_{[\ell]})$.

In terms of this function ν on $E'_{[\ell]}$, we fix, for every $m = 0, 1, 2$, a Lipschitz pillow $\hat{E}'_{\ell_m} \subset B_\infty(E'_{\ell_m}, 1/3)$. By Lemma 7.10, $\mathcal{N}(E'_{\ell_m}) \setminus \hat{E}'_{\ell_m}$ has three components and there exists a unique component $D'_m \subset \mathcal{N}(E'_{\ell_m}) \setminus \hat{E}'_{\ell_m}$ which does not meet $\partial\mathcal{N}(E'_{\ell_m})$ essentially; that is, the intersection $D'_m \cap \mathcal{N}(E'_{\ell_m})$ does not contain $(n-1)$-simplices.

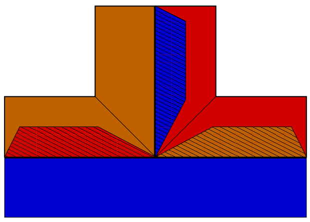

FIGURE 48. Three components D'_m waiting to be connected to corresponding components U_m.

We observe that the set $\bigcup_{m=0}^{2} \hat{E}'_{\ell_m}$ has 6 complementary components in $\mathcal{N}(E'_{[\ell]})$; see Figure 48. We now modify the pillows $\hat{E}'_{\ell_m}$; informally, by connecting each D'_m to U_m, there will be only three complementary components.

For $m = 0, 1, 2$, let $\sigma_{\ell_m} \subset E'_{\ell_m}$ be simplices meeting on a common face $\tau \subset \sigma_{\ell_0} \cap \sigma_{\ell_1} \cap \sigma_{\ell_2}$. By Lemma 6.1, all simplices σ_{ℓ_m} have the same ν-parity. For notational simplicity, we consider only the case $\nu(\sigma_{\ell_m}) = -1$; the case $\nu(\sigma_{\ell_m}) = 1$ is similar and is left to the reader.

We subdivide $\psi_{[\ell]}^{-1}(\tau)$ into congruent subsimplices of side length $1/3$, fix three of these subsimplices and denote by τ_0, τ_1, τ_2 their images under $\psi_{[\ell]}$. Since $\nu(\sigma_{\ell_0}) = \nu(\sigma_{\ell_1}) = -1$, the sheets $\hat\sigma_{\ell_0}$ and $\hat\sigma_{\ell_1}$ of σ_0 and σ_1, respectively, are determined by functions $\Psi_{\sigma_{\ell_0}}$ and $\Psi_{\sigma_{\ell_1}}$. We modify these functions so that

$$\Psi_{\sigma_{\ell_r}}(\mathrm{int}\, \tau_r \times \{2\}) \subset (0, 1/3)$$

for $r = 0, 1$, and denote the new sheets obtained in this manner as $\tilde\sigma_{\ell_r}$ for $r = 0, 1$. We denote also by $\tilde D'_r$ the component of $\mathcal{N}(E'_{\ell_r}) \setminus \tilde\sigma_{\ell_r}$ which does not meet $\partial\mathcal{N}(E'_{\ell_r})$ essentially.

For $r = 0, 1$, let $\tilde U_{k_r}$ be the components of $\mathcal{N}(E'_{\ell_r}) \setminus \tilde\sigma_{\ell_r}$ contained in U_{k_r}. It is now easy to observe that $\tilde D'_r \subset \tilde U_{k_r}$ is connected. Indeed, the $(n-2)$-cell

$$D_r = \{(x,t) \in \tau_r \times \mathbb{R} \colon \Psi_{\sigma_{\ell_r}}(x,1) \le t \le \Psi_{\sigma_{\ell_r}}(x,2)\}$$

for $r = 0, 1$, lies on the boundary of $\tilde D'_r$ and is contained in $\tilde U_{k_r}$. Furthermore, we have that the interior of $\mathrm{cl}(\tilde D'_r \cup U_{k_r})$ is bilipschitz to $\mathbb{B}^n$ in the inner metric.

Without changing notation, we modify the sheet $\hat\sigma_{\ell_2}$ accordingly in order to preserve compatibility with other sheets after this change on $\tau_0 \cup \tau_1$. The sheet modification is now applied to $\hat\sigma_{\ell_2}$ to obtain a new compatible sheet $\tilde\sigma_{\ell_2}$ so that the component D'_2 of $B_\infty(E'_2, 1/3) \setminus \tilde\sigma_{\ell_2}$ is connected to U_{k_2}. We leave the details of this step to the interested reader.

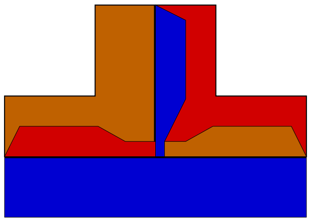

FIGURE 49. Domains after modification; a side view.

We make some observations on the construction of the modified sheets $\tilde\sigma_{\ell_m}$ for $m = 0, 1, 2$. First note that, although $\tilde\sigma_{\ell_m}$ is not homeomorphic to $\hat\sigma_{\ell_m}$ there exist maps $h_{\ell_m} \colon \tilde\sigma_{\ell_m} \to \hat\sigma_{\ell_m}$ so that h_{ℓ_m} is a homeomorphism in the interior of $\tilde\sigma_{\ell_m}$ and $h_{\ell_m}|(\tilde\sigma_{\ell_m} \cap \hat\sigma_{\ell_m}) = \mathrm{id}$. In particular, $\tilde\sigma_{\ell_m}$ has the same number of singular simplices as does $\hat\sigma_{\ell_m}$ and the map h_{ℓ_m} restricts to a map between singular simplices.

Second, let

$$\tilde E'_{[\ell]} = \tilde\sigma_{\ell_0} \cup \tilde\sigma_{\ell_1} \cup \tilde\sigma_{\ell_2}.$$

Then

$$\mathcal{N}(E'_{[\ell]}) \setminus \tilde E'$$

has three connected components $\tilde U_1, \tilde U_2, \tilde U_3$ with the property

$$\partial\tilde U_r \cap \partial U_r = B_\infty(Y, 1/3) \cap \partial U_r.$$

Furthermore, for every $r = 1, 2, 3$, there exists a bilipschitz homeomorphism

$$(\tilde U_r, d_{\tilde U_r}) \to (U_r, d_{U_r}),$$

which is the identity on $\partial\tilde U_r \cap \partial U_r$.

Let

$$\tilde E_{[\ell]} = \psi_{[\ell]}^{-1}(\tilde E'_{[\ell]}).$$

Due to the convention on closed edges on the boundary of $\partial(\bigcup_{r=0}^2 E_{\ell_r})$, we have that

$$\tilde E_{[\ell]} \cap \tilde E_{[\ell']} = E_{[\ell]} \cap E_{[\ell']}$$

for all ℓ and ℓ'.

Write

$$X = \bigcup_{[\ell]} \tilde{E}_{[\ell]},$$

the union over equivalence classes $[\ell]$ of indices. Then $\mathbb{R}^n \setminus X$ has components $\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3$. Using the congruence classes of pillows $\hat{E}'_{[\ell]}$, we may assume that pillows $\tilde{E}'_{[\ell]}$ are uniformly Lipschitz. Then the components $\tilde{\Omega}_1$, $\tilde{\Omega}_2$, and $\tilde{\Omega}_3$ are bilipschitz equivalent to the components Ω_1, Ω_2, and Ω_3 of our original Rickman partition, respectively, in their inner metric. Furthermore, these bilipschitz homeomorphisms $(\Omega_m, d_{\Omega_m}) \to (\tilde{\Omega}_m, d_{\tilde{\Omega}_m})$, $m = 1, 2, 3$, extend to BLD-maps $\mathrm{cl}(\Omega_m) \to \mathrm{cl}(\tilde{\Omega}_m)$. If we set $\tilde{\mathbf{\Omega}} = (\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3)$, then $X = \partial_\cup \tilde{\mathbf{\Omega}}$.

Finally, we obtain a BLD-map $f\colon \partial_\cup \tilde{\mathbf{\Omega}} \to \hat{\mathbb{S}}^{n-1}$. Relabel the components of $\mathbb{R}^n \setminus \hat{\mathbb{S}}^n$ by D_1, D_2, D_3 so that $D_1 = D^U$, $D_2 = D^L$, and $D_3 = D^M$.

By Remark 7.13, we may fix a map $g_{[\ell]}\colon \tilde{E}'_\ell \to \hat{\mathbb{S}}^{n-1}$ as in Lemma 7.12. By Lipschitz uniformity of the pillows $\tilde{E}'_{[\ell]}$, we may assume that $g_{[\ell]}|\mathrm{int}\, \tilde{E}'_{[\ell]}$ is BLD with BLD-constant depending only on n. Let $f_{[\ell]}\colon \tilde{E}_{[\ell]} \to \hat{\mathbb{S}}^{n-1}$, $f_{[\ell]} = g_{[\ell]} \circ \psi_{[\ell']}|\tilde{E}_{[\ell]}$.

Given adjacent pillows $\tilde{E}_{[\ell]}$ and $\tilde{E}_{[\ell']}$, the mappings $g_{[\ell]}$ and $g_{[\ell']} \circ \psi_{[\ell']} \circ \psi_{[\ell]}^{-1}$ are both defined on $\tilde{E}_{[\ell]} \cap \tilde{E}_{[\ell']}$. By uniformity of the BLD-constants, we may modify one of the mappings $g_{[\ell]}$ and $g_{[\ell']}$ slightly to obtain a new collection of uniformly BLD-mappings so that mappings $f_{[\ell]}$ and $f_{[\ell']}$ agree on $\tilde{E}_{[\ell]} \cap \tilde{E}_{[\ell']}$ for every $\ell \neq \ell'$. The map f, defined so that $f|\tilde{E}_{[\ell]} = f_{[\ell]}$, is BLD.

This concludes the proof of Proposition 7.1.

8. Finishing touch

In this section we prove Propositions 1.4 and 1.5. These proofs are slight generalizations of Propositions 5.1 and 7.1. The proof of Proposition 1.5 is a straightforward modification, so we merely indicate the small differences. For Proposition 1.4, we define a particular class of rough Rickman partitions, called *skewed Rickman partitions*, and show that the method to obtain a rough Rickman partition in the proof of Proposition 7.1 may be modified to obtain skewed Rickman partitions. We first consider skewed Rickman partitions and the proof of Proposition 1.4,

For general $p > 2$, choose points $\{y_0, \ldots, y_p\}$ in $\mathbb{S}^n$ as in the introduction, that is, $y_0 = e_{n+1}$ and $y_r = (0, t_r) \in \mathbb{R}^n$, where $-1/2 = t_1 < 0 < t_2 < \cdots < t_p = 1/2$. Take n-cells $E_0, \ldots, E_p$ as in the introduction, i.e. $E_0 = \mathrm{cl}(\mathbb{S}^n \setminus \mathbb{B}^n)$, $E_1 \cup \ldots \cup E_p = \mathbb{B}^n$, $y_r \in \mathrm{int}\, E_r$, so that $B_r = E_r \cap E_{r+1}$ is an $(n-1)$-cell for $r = 0, \ldots p \pmod p$. If $S_r = \partial E_r$, then S_r is an $(n-1)$-sphere, here consisting of $(n-1)$-cells $B_r \cup B_{r-1} \pmod p$. For simplicity, choose $E_1 = \mathbb{B}^n \cap \left(\mathbb{R}^{n-1} \times (-\infty, 0]\right)$ as in Figure 1.

Let

$$\hat{\mathbb{S}}^{n-1}_p = \bigcup_{r=0}^{p} \partial E_r.$$

We emphasize that $E_i \cap E_j = \mathbb{S}^{n-2}$ for $|i-j| > 1$. If

$$\mathcal{E}_p = \{E_1, \ldots, E_{p+1}\},$$

then $\mathcal{E}_p$ is an essential partition of $\mathbb{B}^n$, $\partial_\cup \mathcal{E}_p = \hat{\mathbb{S}}^{n-1}_p$, $\partial_\cap \mathcal{E}_p = \mathbb{S}^{n-2}$ and the adjacency graph $\Gamma(\mathcal{E}_p)$ is cyclic of length of $p+1$.

In fact, we prove a slightly stronger form of Proposition 1.4; note that the case $p = 2$ is covered by Proposition 5.1. Let q be a k-cube. An embedding $\varphi\colon q \to \mathbb{R}^n$ is a *singular k-cube* and a complex composed of singular cubes a *skew complex* if the singular k-cubes are PL and uniformly bilipschitz equivalent. A Rickman partition $\mathbf{\Omega}$ is *skew* if $\partial_\cup \mathbf{\Omega}$ is a skew complex.

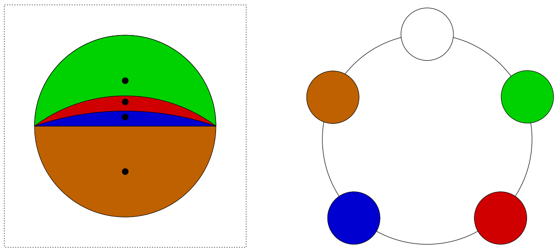

FIGURE 50. The adjacency graph for cells in Figure 1.

Proposition 8.1. *Given $n \geq 3$ and $p > 2$ there exists a skew Rickman partition $\mathbf{\Omega} = (\Omega_0, \ldots, \Omega_p)$ supporting the tripod property.*

Heuristically, the skew Rickman partition $\mathbf{\Omega} = (\Omega_1, \ldots, \Omega_p)$ is obtained by subdividing the third component Ω_3' of a rough Rickman partition $(\Omega_1', \Omega_3', \Omega_3')$ provided by Proposition 5.1 into BLD-half spaces; Figure 51 shows the relations of the domains schematically. We indicate now the essential parts of this construction.

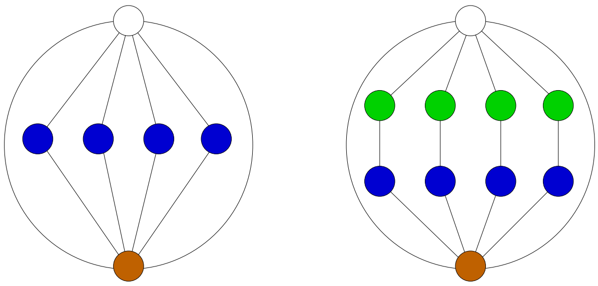

FIGURE 51. Adjacency graphs of components of particular (rough) Rickman partitions for $p = 2$ and $p = 3$; $n = 3$.

8.1. **Skewed structures on atoms and molecules.** Let A be an r-fine $\mathbb{R}^{n-1}$-based atom in $\mathbb{R}^n$; denote $F = A \cap \mathbb{R}^{n-1}$ and $C = \partial A - F$, where 'F' refers to 'floor' and 'C' to 'ceiling'. Note that F and C are $(n-1)$-cells.

Let $\delta_B \colon F \to [0, r/2]$ be the distance function

$$\delta_B(x) = \operatorname{dist}(x, F \cap C).$$

Given $j \in \{2, \ldots, p-1\}$, we denote by L_j the graph of the function $\delta_{B,j} \colon F \to [0, r/3]$ defined by

$$\delta_{B,j}(x) = \frac{2j}{p} \max\{\frac{r}{3}, \delta_B(x)\}$$

for $x \in F$. For notational consistency, set $L_1 = F$ and $L_p = C$.

Then, for every $j = 2, \ldots, p$, $L_{j-1} \cup L_j \cup \partial A$ bounds a unique n-cell A_j with boundary $L_{j-1} \cup L_j$, so that A_j is a skewed atom, called a *skewed copy of A*.

The essential partition

$$\mathcal{S}(A) = (A_2, \ldots, A_p) \tag{8.1}$$

is a *skewed partition of A*. Note that $F \subset \partial A_2$, $C \subset \partial A_p$, and $A_{j-1} \cap A_j$ is the $(n-1)$-cell L_{j-1} for all $j = 3, \ldots, p$. The cells A_j are certainly bilipschitz equivalent. To be specific, let $\pi \colon \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$ be the projection $(x,t) \mapsto x$ and

$$F_{1/3} = \{x \in F \colon \delta_B(x) \geq r/3\},$$

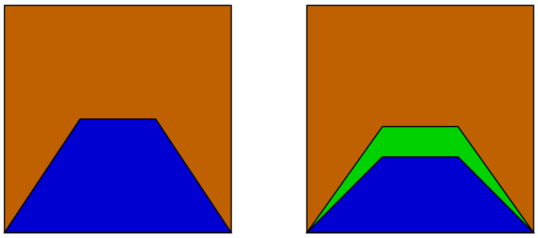

FIGURE 52. Schematic figure on n-cells A_j for $p = 3$ and $p = 4$.

and let $C^\flat$ be the subdivision of the ceiling $C = \partial A - F$ into $(n-1)$-cubes of side length r. We record these observations to the following lemma. We leave the details (again) to the interested reader.

Lemma 8.2. *Let A be an r-fine $\mathbb{R}^{n-1}$-based atom in $\mathbb{R}^n$ and $\mathcal{S}(A) = (A_2, \ldots, A_p)$ a skewed partition of A for $p > 2$. Then there exist L-bilipschitz homeomorphisms $\varphi_j \colon A \to A_j$ for $j = 2, \ldots, p$, where $L = L(n,p)$, so that*

(i) $\varphi_2|F = \mathrm{id}$, $\varphi_p|C = \mathrm{id}$, $\varphi_j|F \cap C = \mathrm{id}$, $(\pi \circ \varphi_j)|F_{1/3} = \mathrm{id}$, *and*
(ii) $\varphi_j(F) = L_{j-1}$ *and* $\varphi_j(C) = L_j$,

where $F = A \cap \mathbb{R}^{n-1}$ and $C = A - F$. Furthermore, these homeomorphisms can be taken to satisfy the additional condition

(iii) $\pi \circ \varphi_{j-1}(c) = \pi \circ \varphi_j(c)$ *for all $j = 3, \ldots, p$ and $c \in C^\flat$.*

Using the homeomorphisms φ_j, $j = 1, \ldots, p$, in Lemma 8.2, set

$$L^\flat_{j+1} = \{\varphi_j(Q) \colon Q \in C^\flat\} = \varphi_j(C^\flat).$$

Clearly, $L^\flat_{j+1}$ is an essential partition of L_{j+1} into $(n-1)$-cells. By (iii) in Lemma 8.2, we also have

$$F^\flat = \{\pi_{j+1}(Q) \colon Q \in L^\flat_{j+1}\} = \pi_{j+1} \circ \varphi_j(C^\flat)$$

for every j and the essential partition $F^\flat$ is well-defined. By the choice of mappings φ_j, we also have

$$L^\flat_j = \varphi_j(F^\flat).$$

In particular, $L^\flat_j \cup L^\flat_{j-1}$ is a skew structure on ∂A_j.

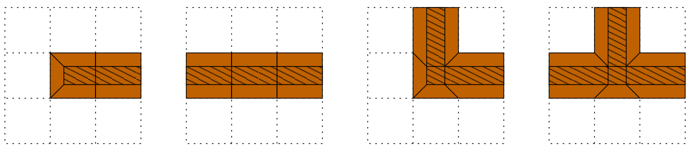

FIGURE 53. Induced cubical structures $F^\flat$ on F for some building blocks.

Lemma 8.2 generalizes easily to molecules.

Lemma 8.3. *There exists $L = L(n,p)$ with the following properties. Let M be a molecule consisting of building blocks on the boundary of an n-cube Q so that pair-wise unions of adjacent building blocks of different finesse are planar. Then there exist an essential partition $\mathcal{S}(M) = (M_2, \ldots, M_p)$ of M into n-cells and L-bilipschitz homeomorphisms $\psi_j \colon M \to M_j$, $j = 2, \ldots, p$, for which*

(a) $\partial M \cap \partial Q \subset \partial M_2$, $\partial M - \partial Q \subset \partial M_p$,
(b) $\psi_i(M) \cap \psi_j(M)$ *is an $(n-1)$-cell if $j = i + 1$, and*
(c) $\psi_i(M) \cap \psi_j(M) = F \cap M$ *for $|i - j| > 1$,*

where $F = M \cap \partial Q \subset \psi_3(M)$.

Proof. It suffices to consider two separate cases: (i) a non-planar atom in $\Gamma(M)$, and (ii) two adjacent atoms in $\Gamma(M)$.

Suppose first that A is a non-planar atom in $\Gamma(M)$. Then A consists of planar parts, all meeting in pairs of building blocks. Thus the general case follows from a special case of building blocks B and B' based on different faces of an n-cube, say Q'. There exists a cube q of side length r in $B \cup B'$ contained in one of the building blocks, say B, so that $q \cap B' = B \cap B'$. Since $A' = B' \cup q$ is an atom, we find skewed atoms A'_j and B_j for $j = 2, \ldots, p$, in A' and B, respectively, so that A'_2 and B_2 meet $\partial Q'$. Since $A'_j \cup B_j$ are n-cells for $j = 2, \ldots, p$, it is now easy to define non-planar n-cells $A_2, \ldots, A_p$ forming an essential partition of A.

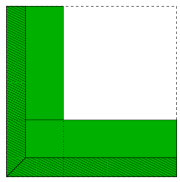

Figure 54. Join of two skewed non-planar building blocks.

Suppose now that A is an r-fine atom adjacent to an $(r/3)$-fine atom A'. Again, there exist building blocks $B \subset A$ and $B' \subset A'$ so that $A' \cap A = B \cap B'$. We may assume that $B \cup B'$ is $\mathbb{R}^{n-1}$-based. Let $\mathcal{S}(B) = (B_2, \ldots, B_p)$ and $\mathcal{S}(B') = (B'_2, \ldots, B'_p)$ be skewed partitions of B and B'. Let $\varphi_j \colon B \to B_j$ and $\varphi'_j \colon B' \to B'_j$ be homeomorphisms as in Lemma 8.2. It is now easy to modify these homeomorphisms on $A \cap A'$ and obtain homeomorphisms $\tilde{\varphi}_j$ and $\tilde{\varphi}'_j$ for $j = 2, \ldots, p$, so that $\tilde{\varphi}_j(B) \cup \tilde{\varphi}_j(B')$ is an n-cell. Since the modification is local, we may also assume that mappings $\tilde{\varphi}_j$ and $\tilde{\varphi}'_j$ are uniformly bilipschitz with a constant depending only on n. We leave the further details to the interested reader; see Figure 55. $\square$

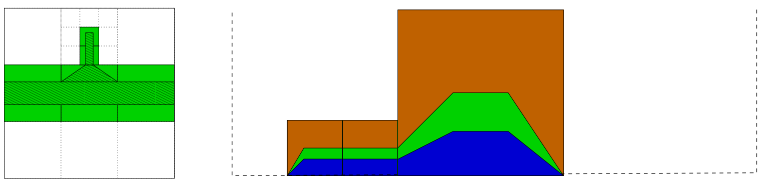

Figure 55. Two building blocks and a common subdivision; $p = 5$.

Since a dented molecule is bilipschitz equivalent to its hull in the sense of Proposition 3.12, the following corollary readily follows.

Corollary 8.4. *Let D be a dented molecule having a hull $M = \mathrm{hull}(D)$ consisting of building blocks so that pair-wise unions of adjacent building blocks are planar. Then D has an essential partition $\mathcal{S}(D) = (D_2, \ldots, D_p)$ with a skew structure.*

Proof. Let $\psi_D \colon M \to D$ be a PL-map as in Proposition 3.12 and let $\mathcal{S}(M) = (M_2, \ldots, M_p)$ be an essential partition as in Lemma 8.3. Then

$$\mathcal{S}(D) = (\psi_D(M_2), \ldots, \psi_D(M_p))$$

is an essential partition of D into PL n-cells L-bilipschitz equivalent to D, where L depends only on n. $\square$

The proof of Proposition 8.1 is now merely a summary on the discussion in this section. Since the details here are analogous to Section 5, we merely sketch the proof and leave the details to the interested reader.

Sketch of proof of Proposition 8.1. Let $\mathbf{\Omega}' = (\Omega'_0, \Omega'_1, \Omega'_2)$ be a rough Rickman partition of $\mathbb{R}^n$ as in the proof of Proposition 5.1 and let C_∞ be one of the components of Ω'_3. We fix an infinite branch Γ in $\Gamma(C_\infty)$ starting from a leaf of $\Gamma(C_\infty)$. Let v_0 be the starting point of Γ and denote by v_m the unique vertex in Γ of distance m to v_0. Let Γ_m be the finite tree containing v_0 and v_m separated by v_{m+1} from $\Gamma(C_\infty)$. Then (Γ_m, v_m) is a rooted tree and $C_m = |\Gamma_m|$ is a dented molecule.

In Γ_m only leaves and vertices adjacent to leaves are atoms, all other vertices are dented atoms. Any two adjacent atoms satisfy the additional planarity condition in Lemma 8.3. By the construction of Proposition 5.1, given an atom A and a hull $A' = \mathrm{hull}(D')$ of a dented atom, a similar planarity condition also holds for M and M'. Hence, analogous to the proof of Lemma 8.3, there is an essential partition $\mathcal{S}(C_m) = (C_{m,2}, \ldots, C_{m,p})$ so that n-cells $C_{m,j}$ are bilipschitz to C_m with a constant depending only on n and p.

Since the construction is performed one dented atom at the time, we may assume that $C_{m+1,j} \cap C_m = C_{m,j}$ for all m and j. Thus $C_{\infty,j} = \bigcup_{m\geq 0} C_{m,j}$ is well-defined and bilipschitz to C_∞. Furthermore, $\mathcal{S}(C_\infty) = (C_{\infty,2}, \ldots, C_{\infty,p})$ carries a skew structure, induced by skew structures on each $\mathcal{S}(C_m)$.

Let $\Omega_0 = \Omega'_0$ and $\Omega_1 = \Omega'_1$. Let also $\Omega_j = \bigcup_{C_\infty} C_{\infty,j}$, where C_∞ is a component of Ω'_3. Then $\mathbf{\Omega} = (\Omega_0, \ldots, \Omega_p)$ is the required skewed Rickman partition. $\square$

8.2. **Proof of Proposition 1.5.** Let $\mathbf{\Omega} = (\Omega_0, \ldots, \Omega_p)$ be a skew Rickman partition as in Proposition 8.1. Then $\partial_\cup \mathbf{\Omega}$ carries a uniformly bilipschitz triangulation into $(n-1)$-simplices together with associated labeling function.

Due to the cyclic combinatorics of domains in $\mathbf{\Omega}$, that is, since $\Omega_j \cap \Omega_{j+1}$ is a punctured $(n-1)$-cell for $j = 0, \ldots, p \pmod p$, we define a parity function $\nu_{\partial_\cup \mathbf{\Omega}} \colon (\partial_\cup \mathbf{\Omega})^{(n-1)} \to \{\pm 1\}$ for $p > 2$ analogous to the case $p = 2$ in Section 6.

To construct a pillow cover over the triangulation of $\partial_\cup \mathbf{\Omega}$ it suffices to discuss only the elementary case of one $(n-1)$-simplex. The pillow cover is then constructed analogously to the case $p = 2$ in Section 7.

Suppose we are given an $(n-1)$-simplex σ in $\mathbb{R}^{n-1}$ which represents a simplex of negative parity. The sheets $\hat\sigma_1, \ldots, \hat\sigma_p$ are graphs of function $\Psi_\sigma \colon \sigma \times \{1, \ldots, p\} \to \mathbb{R}$ so that $\hat\sigma_i = \Psi_\sigma(\sigma \times \{i\})$.

We fix first, for $s = 2, \ldots, p$, a function $u_{\sigma,s} \colon \partial\sigma \to \mathbb{R}$ with the properties that $u_{\sigma,s}|\tau$ is an opening if τ is either an entry or an exit face of σ. Furthermore, we assume that $u_{\sigma,2} < \cdots < u_{\sigma,p}$ if τ is an entry or an exit. Now define Ψ_σ as

(1) $\Psi_\sigma(x, 1) = 0$ and $\Psi_\sigma(x, s) = u_{\sigma,s}(x)$ for $x \in \partial\sigma$, and
(2) $\Psi_\sigma(x, 1) < 0 < \Psi_\sigma(x, 2) < \ldots < \Psi_\sigma(x, p)$ for all $x \in \mathrm{int}\, \sigma$.

Suppose now that σ represents a simplex of positive parity. The sheets $\hat\sigma_i$, for $i = 1, \ldots, p+2$, are now determined by the graph of $\Psi_\sigma \colon \sigma \times \{1, \ldots, p+2\} \to \mathbb{R}$.

Analogously to Section 7.1, first take functions $u_{\sigma,s}$ for $s = 2, \ldots, p+1$ so that $u_{\sigma,s}|\tau$ is an opening if τ is either an entry or exit face, and $u_{\sigma,s}|\tau = 0$ otherwise. We also assume that $u_{\sigma,2} < \cdots < u_{\sigma,p+1}$.

To obtain functions $v_{\sigma,s} \colon \partial\sigma \to \mathbb{R}$ for $s = 2, \ldots, p+1$, let τ_σ be the unique exit face of σ. We then find functions $v_{\sigma,s}$ for which $v_{\sigma,s}|\tau_\sigma$ is a shuffle and $v_{\sigma,s}|\tau = 0$ for other faces τ of σ. Furthermore, using the notation from Section 7.1, we arrange that $v_{\sigma,2}|\mathrm{int}\ \tau_0 < \cdots < v_{\sigma,p+1}|\mathrm{int}\ \tau_0$ and $v_{\sigma,s}|(\tau_i \cup \tau_j) = v_{\sigma,2}|(\tau_i \cup \tau_j)$ for all $s = 2, \ldots, p+1$.

As in Section 7.1, we fix a function $\Psi_\sigma \colon \sigma \times \{1, \ldots, p+2\} \to \mathbb{R}$ satisfying the following conditions

(1) $\Psi_\sigma(x,1)=\Psi_\sigma(x,2)=0$, $\Psi_\sigma(x,s)=u_{\sigma,s}(x)$, and $\Psi_\sigma(x,p{+}2)=\Psi_\sigma(x,p{+}1)$ for $s=2,\dots,p+1$ if $v_{\sigma,2}(x)=0$,
(2) $\Psi_\sigma(x,1)=\Psi_\sigma(x,2)$, $\Psi_\sigma(x,s)=v_{\sigma,s}(x)$, and $\Psi_\sigma(x,p+2)=0$ for $s=2,\dots,p+1$ if $v_{\sigma,2}(x)<0$,
(3) $\Psi_\sigma(x,1)=0$, $\Psi_\sigma(x,s)=v_{\sigma,s}(x)$ for $s=2,\dots,p+1$, and $\Psi_\sigma(x,p+2)=\Psi_\sigma(x,p+1)$ if $v_{\sigma,2}(x)>0$.

To illustrate the effect of this 'shuffle', suppose σ and σ' are adjacent simplices in $\mathbb{R}^{n-1}$ with parities $+1$ and -1, respectively. Let $U=(\sigma\times\mathbb{R})\setminus\hat\sigma$, $V=(\sigma'\times\mathbb{R})\setminus\hat\sigma'$ and $W=((\sigma\cup\sigma')\times\mathbb{R})\setminus(\hat\sigma\cup\hat\sigma')$. The closures of components of U, V, and W have natural adjacency graphs $\Gamma(U)$, $\Gamma(V)$, and $\Gamma(W)$ illustrated in Figure 56; compare to [13, Fig. 8.4].

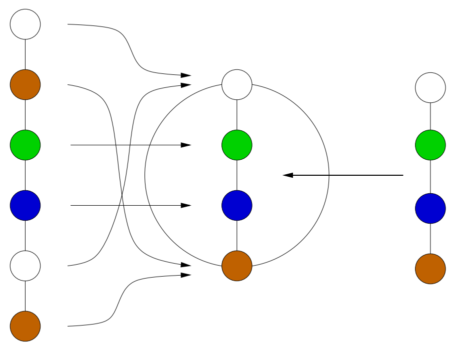

FIGURE 56. Adjacency graphs $\Gamma(U)$, $\Gamma(V)$ and $\Gamma(W)$ and maps of graphs induced by inclusions $U\hookrightarrow W\hookleftarrow V$; $p=3$.

Having these pillows at our disposal, we may follow the general pillow cover construction in Section 7 almost verbatim, and thus obtain a Rickman partition $\widetilde{\mathbf{\Omega}}=(\widetilde\Omega_1,\dots,\widetilde\Omega_p)$ and a BLD-map $f\colon \partial_\cup\widetilde{\mathbf{\Omega}}\to\hat{\mathbb{S}}^{n-1}_p$. This concludes the proof of Proposition 1.5.

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